Geel 2000 Language Schools

Math Department

Second Term

Sec. 1

2024/2025

Name :-----------------------------------

Class:-----------------------------------


ALGEBRA


MATRIX
at
pppst.com
$\begin{bmatrix} 2x \\ 2x + 3y \end{bmatrix} = \begin{bmatrix} y \\ 12 \end{bmatrix}$

Lesson (1) : Matrices

The matrix: "Is an organization of some elements written in rows and columns between brackets in the form ( ) ".

Ex:

$$\begin{array}{cl} \begin{array}{ccc} 1^{st}\text{ column} & 2^{nd} & 3^{rd} \end{array} & \\ \begin{pmatrix} -5 & 3 & 10 \\ 1 & 4 & -4 \\ 0 & \sqrt{3} & 7 \end{pmatrix} & \begin{array}{l} \rightarrow 1^{st}\text{ row} \\ \rightarrow 2^{nd}\text{ row} \\ \rightarrow 3^{rd}\text{ row} \end{array} \end{array}$$

The order of any matrix = no. of rows x no. of columns

How to express the elements in the matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

a_{32}

row column

$\therefore$ a_{32} is the element in 3^{rd} row and the 2^{nd} column.

Some types of matrices:

A **Square matrix:** It is a matrix in which the number of its rows equals the number of its columns. For example: $\begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$ (a 2×2 square matrix)

B **Row matrix:** It is a matrix containing one row and any number of columns. For example : $\begin{pmatrix} 2 & 4 & 6 & 8 \end{pmatrix}$ (a 1×4 row matrix)

C **Column matrix:** It is a matrix containing one column and any number of rows. For example: $\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$(a 3×1 column matrix)

D **Zero matrix:** It is a matrix in which all of its elements are Zeros. It may be a square matrix or not. For examples:

(0) is a 1×1 zero matrix, $(0 \quad 0)$ is a 1×2 zero matrix, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a 2×1 zero matrix, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a 2×2 square zero matrix and is denoted by $\bigcirc$.

E **Diagonal matrix:** It is a square matrix in which all elements are zeros except the elements of its diagonal then at least one of them is not equal to zero. For example: the matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (is a 3×3 diagonal matrix)

F **Unit matrix:** it is a diagonal matrix in which each element on the main diagonal has the numeral 1, while 0 exists in all other elements , it is denoted by I. for example: each of:

(1) , $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a unit matrix.

Transpose of matrix:

If $A = (a_{xy})$ then $A^T = a_{yx}$

Where A^T is the transpose of A

Note: $(A^T)^T = A$

Ex1: Write the matrix (A_{xy}) of the dimensions 3 × 2 where: $a_{xy} = 2x - y$

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Ex2: Write the matrix (B_{xy}) of the order 3 × 3 where: $b_{xy} = 3x - 2y$

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Ex3: Find the transpose of the following matrices and write its order:

$A \begin{pmatrix} 2 & 3 & 0 \\ -1 & 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 9 \\ -2 \\ 4 \end{pmatrix}$ and $C = \begin{pmatrix} -7 & 5 \\ 9 & 4 \end{pmatrix}$

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The equality of two matrices

If A and B are two matrices then A = B if and only if

1- A and B with the same order

2- The corresponding elements are equal.

$$\begin{pmatrix} 1 & 0 & -2 \\ 2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 3 & -1 \end{pmatrix}$$

Ex1: Find the values of x, y and Z if

$$\begin{pmatrix} 7 & 0 & 2 \\ 4 & 7 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 0 & X+5 \\ 4 & 2y-3 & 5 \end{pmatrix}$$

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Ex2: If X = $\begin{pmatrix} 3a+1 & 12-b & h^3 \\ c+2d & 18 & 6 \end{pmatrix}$ Y = $\begin{pmatrix} 1 & 9 \\ 3 & 18 \\ -8 & d+2c \end{pmatrix}$

Find a , b , c , d and h if X = Y^T

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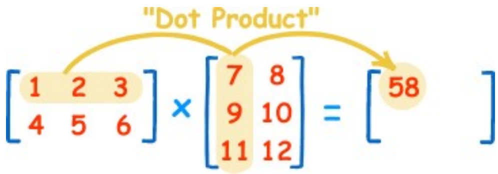
Symmetric and skew symmetric matrices:

If A is a square matrix , then

- A is called a symmetric matrix if and only if $A = A^T$
- A is called a skew symmetric matrix if and only if $A = - A^T$

$A = \begin{pmatrix} 2 & -1 & -3 \\ -1 & 4 & 0 \\ -3 & 0 & 5 \end{pmatrix}$ is symmetric matrix $B = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & \frac{1}{2} \\ 2 & -\frac{1}{2} & 0 \end{pmatrix}$ is skew symmetric



sheet (1)

Choose the correct answer from those given :

(1) If $A = \begin{pmatrix} 1 & 1 & x-1 \\ 1 & 3 & 5 \\ -1 & 5 & 6 \end{pmatrix}$ is a symmetric matrix , then $x =$ ……………

(a) -1 (b) zero (c) 4 (d) 6

(2) If $A = \begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 \\ \frac{1}{2}k & 4 \end{pmatrix}$ where $A = B^t$, then $k =$ ……………

(a) -2 (b) $-\frac{3}{2}$ (c) 8 (d) -6

(3) If $A = \begin{pmatrix} 1 & 5 \\ 3 & 2 \\ -1 & 7 \end{pmatrix}$, then $a_{12} + a_{32} =$ ……………

(a) 8 (b) 12 (c) zero (d) 10

(4) If $\begin{pmatrix} 1 & x & 2 \\ -1 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & 6 \\ 2 & y \end{pmatrix}^t$, then $x\,y =$ ……………

(a) -15 (b) -2 (c) 2 (d) 15

Complete the following :

(1) If A is a matrix of order 2×2 and if $a_{11} = 3$, $a_{12} = 5$, $a_{21} = \frac{1}{2}$ and $a_{22} = \sqrt{5}$, then the matrix A = ……………

(2) If A is a matrix of order 3×2 and if $a_{11} = 2$, $a_{21} = 3$, $a_{32} = \frac{1}{2}\, a_{11}$, $a_{22} = a_{21} + 3$, $a_{31} = -9$, $a_{12} = \frac{1}{3}\, a_{31}$, then the matrix A = ……………

(3) If $Y = \begin{pmatrix} -4 & 2 & 5 \\ 1 & -\sqrt{3} & 9 \end{pmatrix}$, then the matrix Y is of order ……………

, $y_{21} =$ …………… , $y_{22} =$ …………… , $\frac{1}{2}\, y_{12} + \sqrt{y_{23}} =$ ……………

(4) If A is a matrix of order 2×3 , then the number of elements of the matrix A is ……………

(5) If B is a matrix of order 3×1 , then B^t is a matrix of order ……………

(6) If O is a zero matrix of order 3×3 , then $O^t =$ ……………

3] If A = $\begin{pmatrix} 5 & 2x & 8 \\ -4 & -3 & 6 \\ x+2y & 6 & 4 \end{pmatrix}$ is a symmetric matrix , then Find the value of : x , y

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4] If B = $\begin{pmatrix} 0 & 3x & 7 \\ x+3 & 0 & -2z \\ 3y-x & 6 & 0 \end{pmatrix}$ is skew symmetric matrix Find the value of x , y and z

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Lesson (2) : Operation on matrices

I-Addition and subtraction:

To add two matrices A, B they must have the same order.

Ex1: If A = $\begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$, B = $\begin{pmatrix} 6 & -7 \\ 4 & 3 \end{pmatrix}$

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Ex2: If A = $\begin{pmatrix} 2 & -2 \\ 4 & 6 \end{pmatrix}$, Find 3A

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Ex3: If A $\begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$, B = $\begin{pmatrix} -1 & 3 \\ 4 & 6 \end{pmatrix}$, C = $\begin{pmatrix} 0 & 5 \\ -2 & 4 \end{pmatrix}$

Find: (1) A + B (2) B – C (3) A + 2B – C

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Sheet (2)

I-Complete:

1) $IA + \begin{pmatrix} -3 & -2 \\ 5 & 4 \end{pmatrix} = 0$, then A = ……………………………

2) If O is the Zero matrix of order 2 x 2 , then 4O = ……………and it is of order………………………

3) If each of the matrices A and B is of order 3 x 1, then the resultant matrix of A – 2B is of order……………………

4) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^T =$ ………………… which is of order……………

5) If $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, then 3 A = ………………, - 2A = ……………

6) If $A = \begin{pmatrix} 15 & 10 \\ 5 & 20 \end{pmatrix}$, then $A = 5 \begin{pmatrix} \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots \end{pmatrix}$

2] If $A = \begin{pmatrix} -3 & 1 \\ 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$

Check that: 1) $(A + B)^T = A^T + B^T$ 2) $A - B \neq B - A$

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3] If $\begin{pmatrix} 3 & 6 \\ 5 & -7 \end{pmatrix} + \begin{pmatrix} 1 & -4 \\ -2 & 6 \end{pmatrix} = \begin{pmatrix} X & 4 \\ 7 & Y \end{pmatrix}$

Find the value of X and Y.

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4] Find X,Y, Z, and L that satisfy that:

$X\begin{pmatrix} 1 & 3 \\ 5 & Y \end{pmatrix} + Z\begin{pmatrix} 2 & L \\ 0 & 4 \end{pmatrix} + 5\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = O_{2x2}$

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7] If $A = \begin{pmatrix} 5 & -3 & 6 \\ 2 & 5 & 0 \\ 4 & -2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 1 & 3 \\ -4 & 21 & -5 \\ 3 & 12 & 6 \end{pmatrix}$

Find the matrix X such that: $3A + X = 2B$

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8] If $X + 2X^T = \begin{pmatrix} 9 & 14 \\ 13 & 6 \end{pmatrix}$, find the matrix X.

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9] If $A = \begin{pmatrix} 2 & -1 \\ 5 & -3 \end{pmatrix}$ and $B^T = \begin{pmatrix} -1 & 0 \\ 3 & 4 \end{pmatrix}$,

Find the matrix X such that: $4X - 3B + 2A^T = A + (5B - X)^T$.

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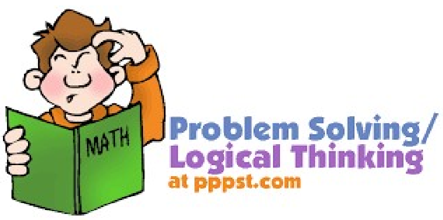

Lesson (3) : Multiplying Matrices

- If A is a matrix of order m×n, B is a matrix of order r × L, then their product C = A x B will be defined if and only if n = r
- To multiply two matrices A no. of columns= no. or rows B

2×3 3×1

2x1 is the order of the product matrix

Ex1: If A= $\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$, B = $\begin{pmatrix} -2 & 1 \\ 5 & 6 \end{pmatrix}$

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Ex2: If A $\begin{pmatrix} 3 & -2 \\ 0 & 2 \\ -1 & 4 \end{pmatrix}$, B = $\begin{pmatrix} 3 & -1 \\ 5 & 7 \end{pmatrix}$, C = $\begin{pmatrix} 4 & 0 & 3 \\ 5 & 2 & -1 \end{pmatrix}$

and D = $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ Check that: 1) $(AB)^T = B^T A^T$ 2) (AB)C = A(BC)……………………………………………………………………………………………
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Sheet (3)

1 Complete the following :

(1) If A is a matrix of order $m \times n$ and B is a matrix of order $r \times \ell$
, then AB is defined if …………… and AB is undefined if ……………

(2) If A is a matrix of order 3×1 and B is a matrix of order 1×3
, then AB is a matrix of order …………… and BA is a matrix of order ……………

(3) If A is a matrix of order 2×3 and AB is defined as a matrix of order 2×1
, then B is a matrix of order ……………

(4) If A is a matrix of order 2×3 and B^t is a matrix or order 1×3
, then AB is a matrix of order ……………

(5) If A is a square matrix , I is the identity matrix of the same order of A ,
then $A \times I = I \times A =$ …………… , $I^t =$ …………… , $I^2 =$ ……………
, $I^3 =$ …………… , $I^n =$ …………… where n is a positive integer.

2] If $X = \begin{pmatrix} -1 & 2 \\ 0 & 3 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, **prove that** : $XY \neq YX$

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3] If $X = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ and $Y = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$, **find** : $X^2 - Y^2$

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Lesson (4) : Determinants

second order

$$|A|=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-cb$$

Ex:1 Find the value of the following determinant :

a) $\begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix}$ b) $\begin{vmatrix} 4 & -7 \\ 2 & 6 \end{vmatrix}$ c) $\begin{vmatrix} 5 & 4 \\ -3 & -2 \end{vmatrix}$ d) $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$

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- Third order

$$|A|=\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix}=a\begin{vmatrix} e & f \\ h & j \end{vmatrix}-b\begin{vmatrix} d & f \\ g & j \end{vmatrix}+c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$=-a\begin{vmatrix} e & f \\ h & j \end{vmatrix}+b\begin{vmatrix} d & f \\ g & j \end{vmatrix}-c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Ex:2 Find the value of the following determinant :

a) $\begin{vmatrix} 4 & -1 & 3 \\ 0 & 5 & -2 \\ 0 & -3 & -1 \end{vmatrix}$ b) $\begin{vmatrix} -1 & 2 & 5 \\ 0 & 3 & -2 \\ 0 & 0 & 6 \end{vmatrix}$

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☐ Another method

$$\begin{vmatrix} a & b & c \\ d & e & l \\ m & n & k \end{vmatrix} \xrightarrow{\text{Repeat the first two}} \begin{vmatrix} a & b & c \\ d & e & l \\ m & n & k \end{vmatrix} \begin{matrix} a & b \\ d & e \\ m & n \end{matrix}$$

S1 = aek + blm + cdn
S2 = bdk + aln + cem
Then the value of the determinant is S = S1 – S2

- **Remark :**

(1) The triangular matrix:

It is a square matrix in which elements above or below principal diagonal are zeroes

Ex) $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$, $\begin{pmatrix} a & b & c \\ 0 & e & l \\ 0 & 0 & k \end{pmatrix}$

Its determinant $= \begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11} \times a_{22}$

And $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11} \times a_{22} \times a_{33}$

(2) Finding the area of triangle using determinants:

If ΔABC ***in which*** $A(x_1,y_1), B(x_2,y_2)$ ***and*** $C(x_3,y_3)$

Then the area of triangle ABC $= \frac{1}{2}|A|$ where $A = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Steps:

a) Find $A = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

b) Area $= \frac{1}{2}\,|A|$

Note: use elements of the 3rd column because it is easier

(3) To prove that three points are collinear:

The three points $(x_1,y1)$,(x_2,y_2) *and* $C(x_3,y_3)$ are collinear if

$$A = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \text{zero}$$

- *Cramer's rule*

First: solving a system of linear equations of two variables:

To solve the two equations $ax + by = m$ and $cx + dy = n$ follow the steps:

1) Find the three determinants Δ, Δx and Δy where

$$\Delta = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \Delta x = \begin{vmatrix} m & b \\ n & d \end{vmatrix}, \Delta y = \begin{vmatrix} a & m \\ c & n \end{vmatrix}, \Delta \neq 0$$

2) To find the value of x, y

$$x = \frac{\Delta_x}{\Delta},\ y = \frac{\Delta_y}{\Delta}$$

Note: If $\Delta = 0$ then the system has no solution

Second: solving a system of linear equations of three variables:

To solve the two equations $a_1x + b_1y + c_1z = m$, $a_2x + b_2y + c_2z = n$ and $a_3x + b_3y + c_3z = k$ follow the steps:

1) Find the four determinants Δ, Δx , Δy and Δz where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta x = \begin{vmatrix} m & b_1 & c_1 \\ n & b_2 & c_2 \\ k & b_3 & c_3 \end{vmatrix}, \Delta y = \begin{vmatrix} a_1 & m & c_1 \\ a_2 & n & c_2 \\ a_3 & k & c_3 \end{vmatrix}$$

$$\Delta z = \begin{vmatrix} a_1 & b_1 & m \\ a_2 & b_2 & n \\ a_3 & b_3 & k \end{vmatrix}, \Delta \neq 0$$

2) To find the value of x, y and z

$$x = \frac{\Delta_x}{\Delta},\ y = \frac{\Delta_y}{\Delta}, z = \frac{\Delta_z}{\Delta}$$

Ex:3 solve the equation : $\begin{vmatrix} x & 0 & 1 \\ 8 & 1-x & -x \\ x & -1 & 1+x \end{vmatrix} = 0$

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Ex:4 Find the area of a triangle whose vertices are

X(1,2) ,Y(3,-4) and Z(-2,3)

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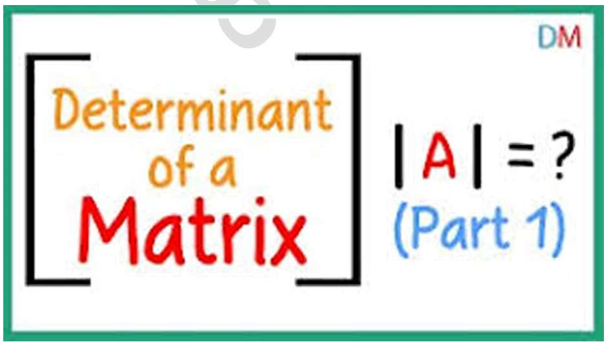

Sheet 4

1 Find the value of each of the following determinants :

(1) $\begin{vmatrix} 7 & 5 \\ 3 & 2 \end{vmatrix}$ (2) $\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$ (3) $\begin{vmatrix} -2 & -2 \\ 4 & 0 \end{vmatrix}$

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2 Prove that :

(1) $\begin{vmatrix} 2x & -1 \\ 2 & 3x \end{vmatrix} + \begin{vmatrix} 3 & 6x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 13 \\ -2 & -7 \end{vmatrix}$

(2) $\begin{vmatrix} \csc\theta & \cot^2\theta \\ 1 & \csc\theta \end{vmatrix} \times \begin{vmatrix} 2 & -3 \\ 5 & -7 \end{vmatrix} = 1$

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3 Find the value of each of the following determinants

(1) $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 4 & 4 \\ 0 & 7 & 8 \end{vmatrix}$ (2) $\begin{vmatrix} 0 & 42 & 3 \\ 2 & 18 & 7 \\ 0 & 28 & 3 \end{vmatrix}$

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4 Solve each of the following equations

(1) $\begin{vmatrix} 2 & 1 \\ 4 & x \end{vmatrix} = 0$

(2) $\begin{vmatrix} x & -1 \\ 2 & x \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 2 & x \end{vmatrix} = 2$

(3) $\begin{vmatrix} 0 & -1 & x \\ x & 4 & 3 \\ 2 & 1 & 2 \end{vmatrix} = 10$

Find using determinants the area of the triangle :

(1) A (2 , 4) , B (– 2 , 4) , C (0 , – 2)

(2) X (3 , 3) , Y (– 4 , 2) , Z (1 , – 4)

Use determinants to prove that each of the following points are collinear :

(1) (3 , 5) , (4 , – 1) , (5 , – 7)

(2) (3 , 2) , (– 1 , 0) , (– 5 , – 2)

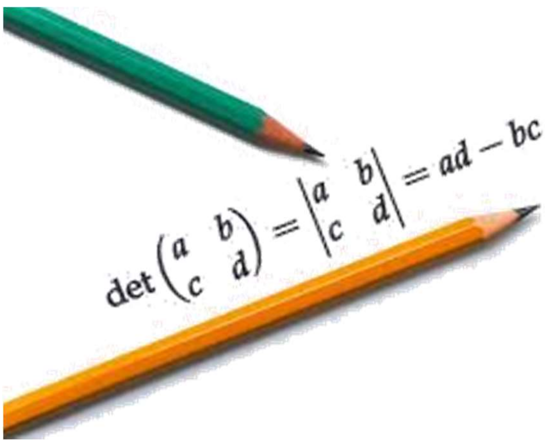
Solve each of the following systems of linear equations by Cramer's rule :

(1) $2x - 3y = 5 \quad , \quad 3x + 4y = -1$

(2) $x + 3y = 5 \quad , \quad 2x + 5v = 8$

Solve each of the following systems of linear equations by Cramer's rule :

(1) $2x + y - 2z = 10 \quad , \quad 3x + 2y + 2z = 1 \quad , \quad 5x + 4y + 3z = 4$



Lesson (5) : Multiplicative inverse of a matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Then $A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $AA^{-1} = A^{-1}A = I$

$\Delta \neq 0$

1] Show the matrix which have multiplicative inverse :

a) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ c) $\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}$

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d) $\begin{pmatrix} 2 & 6 \\ -1 & 3 \end{pmatrix}$ e) $\begin{pmatrix} -1 & 0 \\ 3 & 4 \end{pmatrix}$ f) $\begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix}$

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2] what is the real values of a which make each of the following matrices has A multiplicative inverse :

a) $\begin{pmatrix} a & 1 \\ 6 & 3 \end{pmatrix}$ b) $\begin{pmatrix} a & 9 \\ 4 & a \end{pmatrix}$

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3] if : X = $\begin{pmatrix} 1 & x \\ 0 & -x \end{pmatrix}$ prove that : $X^{-1} = X$

4] solve each of the following system using the matrices :

a) $3x+2y=5$, $2x+y=3$

b) $2x-7y=3$, $x-3y=2$

Sheet 5

1 Show the matrices which have multiplicative inverse and the matrices which have not multiplicative inverse in the following , and find it if it is existed :

(1) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ | (2) $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ | (3) $\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}$

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2] Find the real values of x which make the matrix $\begin{pmatrix} x & 27 \\ 3 & x \end{pmatrix}$ have no multiplicative inverse.

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3] If $X = \begin{pmatrix} 1 & x \\ 0 & -1 \end{pmatrix}$, **prove that :** $X^{-1} = X$

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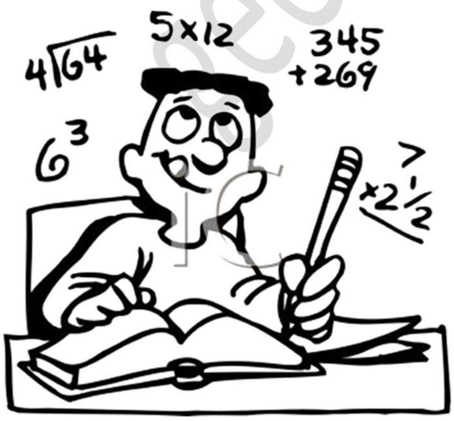
....
....

4] If $A = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$ and $AB = \begin{pmatrix} 4 & -2 \\ 0 & 7 \end{pmatrix}$, **find the matrix B**

5] **Solve each system of the following linear equations using the matrices :**

(1) $3x + 2y = 5 \,,\, 2x + y = 3$

(2) $2x - 7y = 3 \,,\, x - 3y = 2$




LINEAR PROGRAMMING


These students are smart. They'll get it!
Alright just sit back and LEARN!
Applications of Systems of Linear Equations and Inequalities in Real Life
1. System of Linear Equation
2. System of Linear Inequality
Math

Lesson (1): linear inequality

First: Inequality of first degree in one variable

Example

1. Find the solution set of each of the following inequalities where $x \in R$ then represent the solution on the number line:

A $3x - 9 > 6x$

B $6 + x < 3x + 2 \leqslant 14 + x$

Solution

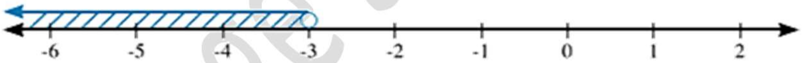
A $3x - 9 > 6x$ add (9 - 6x) to both sides.

$\therefore \; 3x - 9 + 9 - 6x > 6x + 9 - 6x$

$\therefore \; -3x > 9$ (multiply both sides by $-\frac{1}{3}$)

$x < -3$

the solution set = $]-\infty, -3[$



B Divide the inequality into two inequalities as follows:

The first inequality: $6 + x < 3x + 2$

$\therefore 6 - 2 < 3x - x$

$\therefore x > 2$

The solution set = $]2, \infty[$

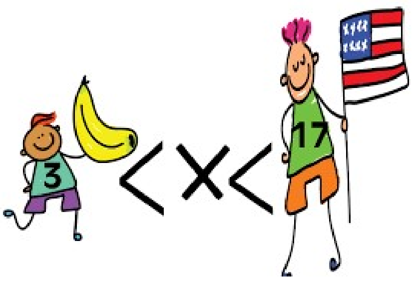
The second inequality: $3x + 2 \leqslant 14 + x$

$\therefore 3x - x \leqslant 14 - 2$

$\therefore \; x \leqslant 6$

The solution set = $]-\infty, 6]$

The solution set = $]2, \infty[\cap]-\infty, 6] =]2, 6]$



Second : Inequality of first degree in two variables

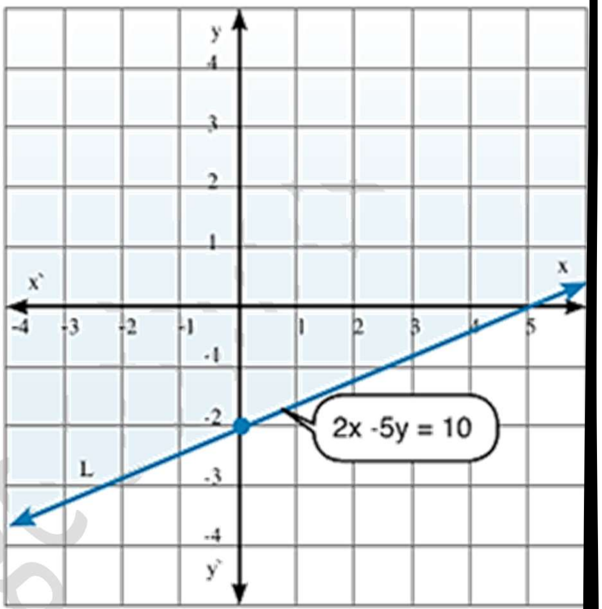
Example

Represent graphically the solution set of the inequality: $2x - 5y \leqslant 10$

Solution

Step (1): represent graphically the boundary line (L). $2x - 5y = 10$ by a solid line (because the inequality relation $\leqslant$).

x	0	5	$2\frac{1}{2}$
y	-2	0	-1



You can draw the boundary line, write the straight line: $2x - 5y = 10$ in the form: $y = mx + c$ where m is the slop and c is the y - intercept from the y-axis.

then: $-5y = -2x + 10 \quad \therefore\ y = \frac{2}{5}x - 2$

Step (2): test the point (0,0) which lies on one side of the boundary line.

$2x - 5y \leqslant 10$ (the original inequality)

$2(0) - 5(0) \overset{?}{\leqslant} 10$ (substitute the point (0, 0))

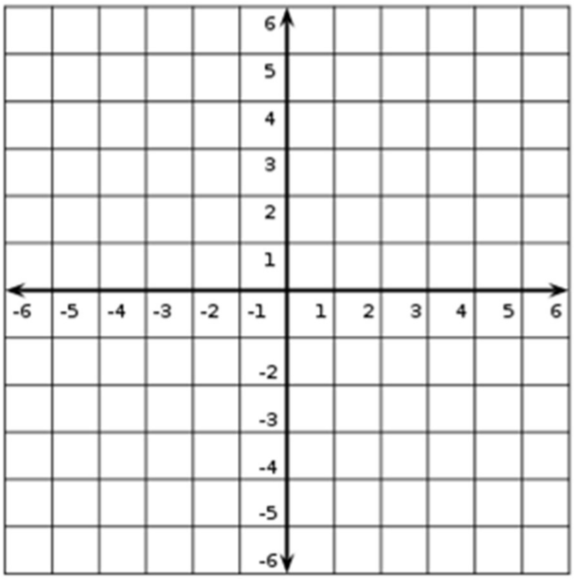
$0 \leqslant 10$ (True)

Colour the region which contains the point (0, 0), where the solution set is half the plane which the point (0, 0) lies ∪ the set of points on the boundary line L.

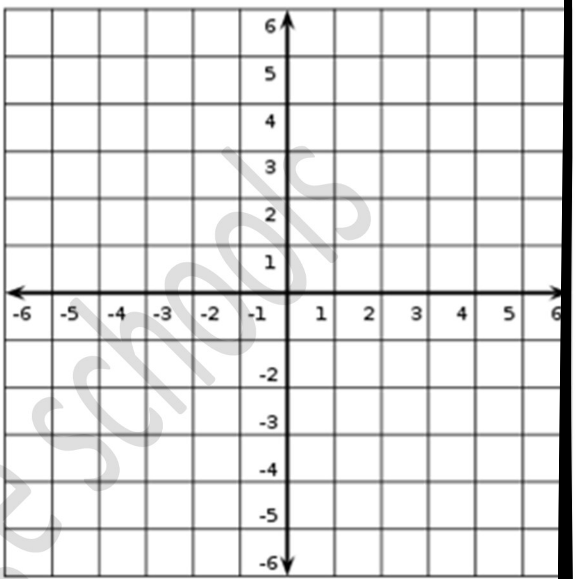
Sheet (1)

(1) Find graphically the S.S of each of the following :

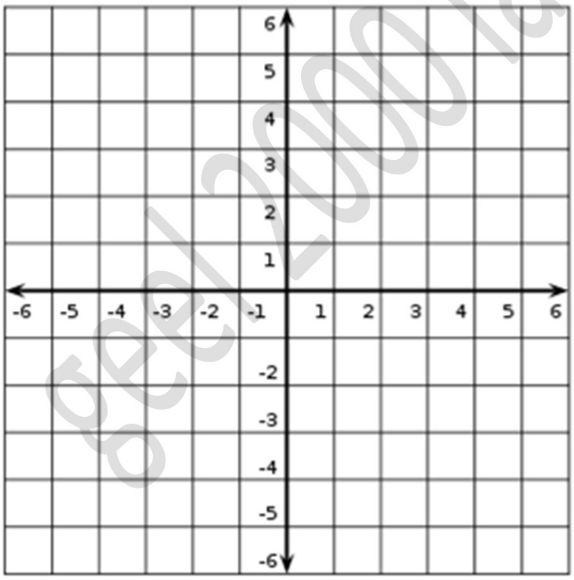
a) $x \geq -2$



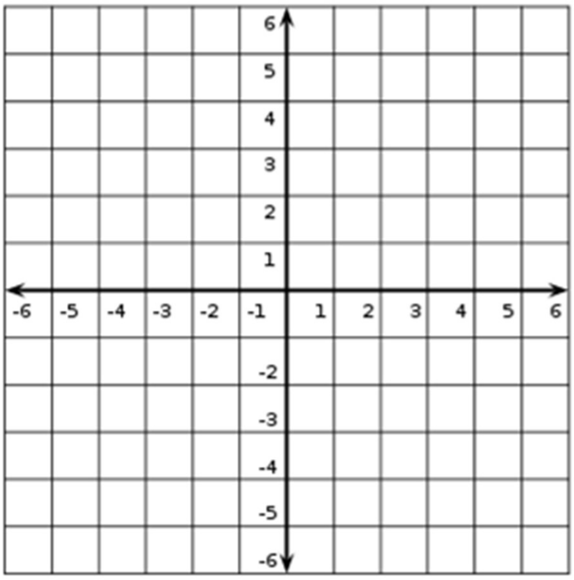
b) $y < 5$



c) $-1 < x \leq 3$



d) $0 \leq y \leq 4$



Solving system of linear inequalities graphically

To solve two or more linear inequalities graphically do the following steps:

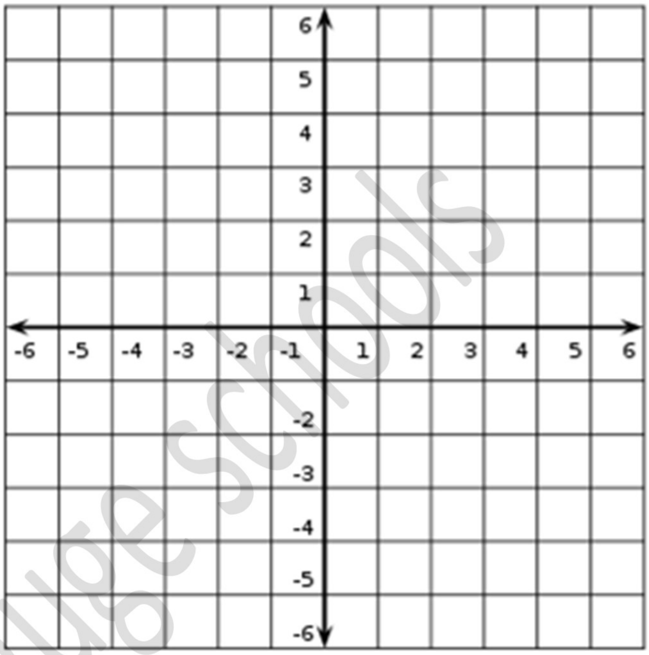
1) Shade the region S_1 that represents the S.S of the 1^{st} inequality.
2) Shade the region S_2 that represents the S.S of the 2^{nd} inequality.
3) The common region S of the two regions S_1 and S_2 represents the S.S of the two inequalities where: $S = S_1 \cap S_2$ as the opposite figure:

Very important remarks:

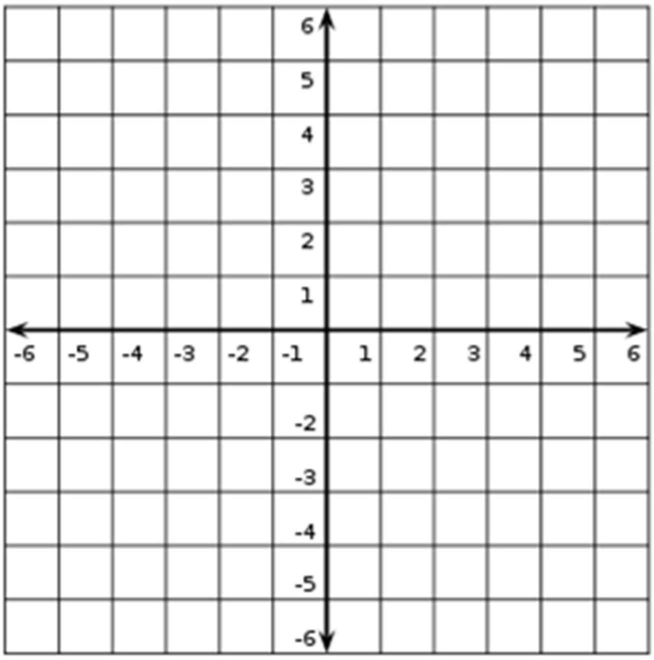
- The eqn. $y = 0$ is represented by X- axis
- The eqn. $X = 0$ is represented by y- axis

(1) Solve each of the following systems graphically:

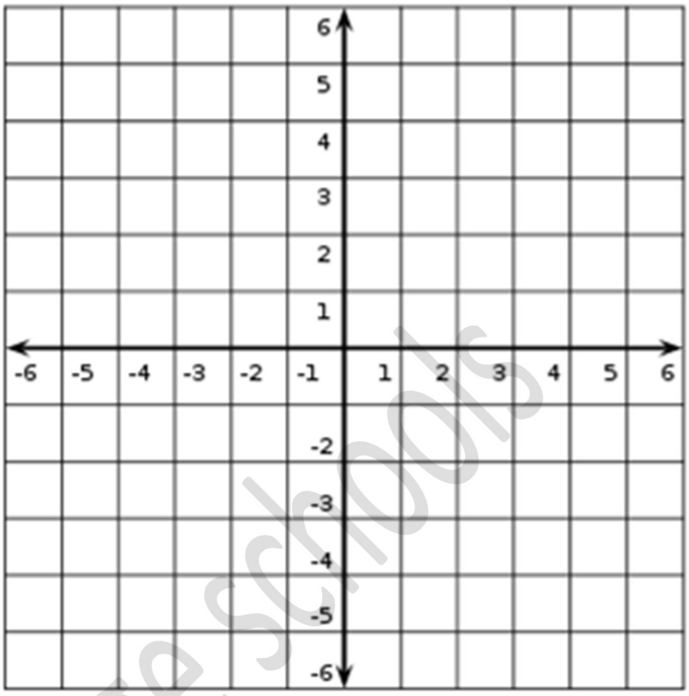
a) $x - 1 > 0$, $y \leq -2$



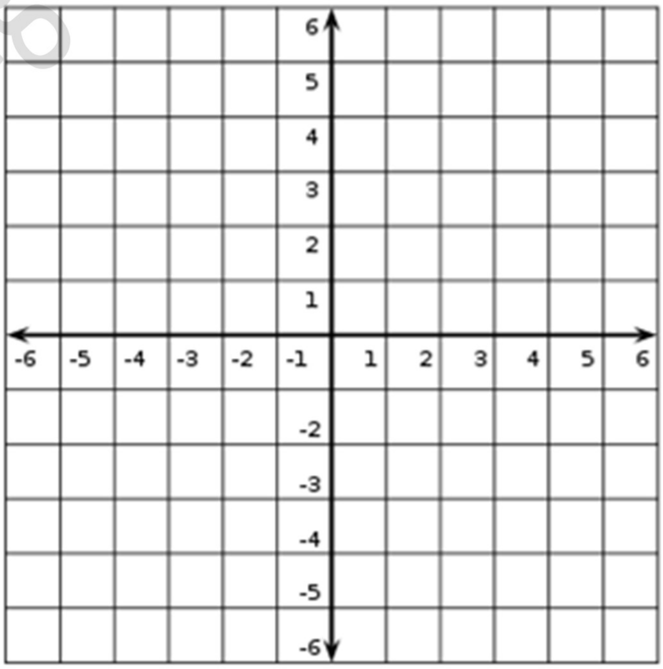
b) $x \geq 0$, $y - 2x < 3$



c) $-2 < x \leq 1 \quad , \ 1 \leq y < 5$



d) $x - 3y \geq 1 \quad , \quad 6y \geq 2 + 2x$



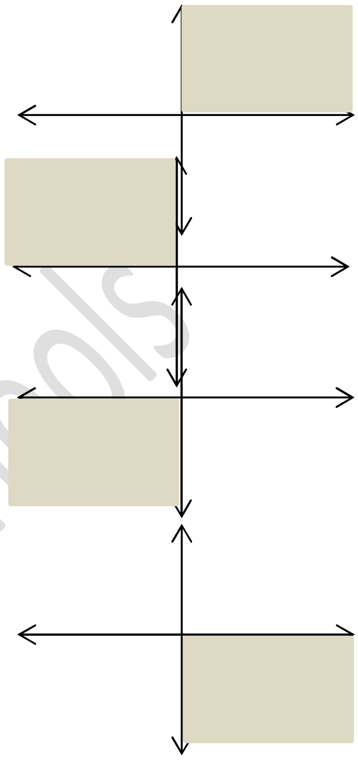
Remarks:

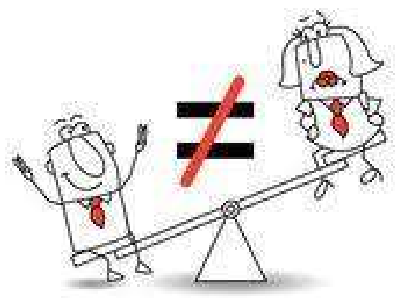
☐ $x \geq 0 , y \geq 0$ represents the 1st quadrant

☐ $x \leq 0 , y \geq 0$ represents the 2nd quadrant

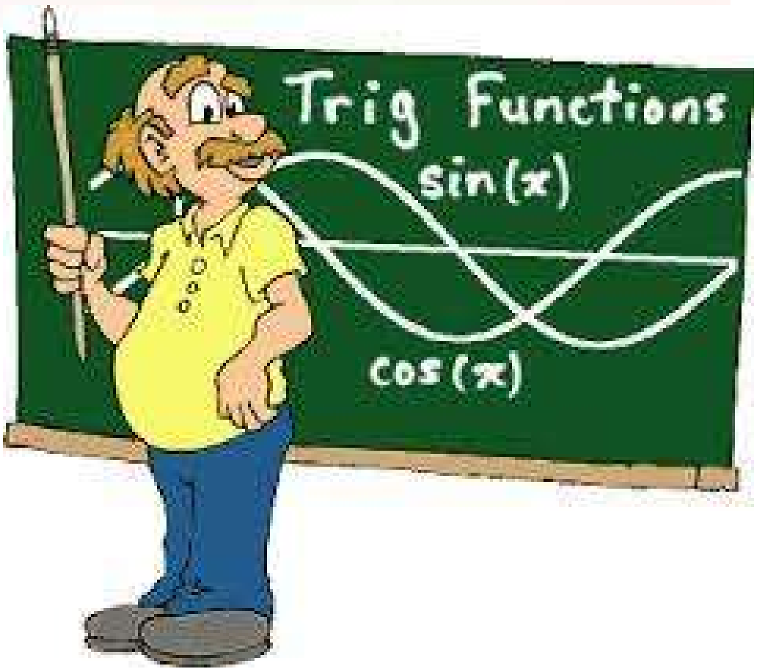
☐ $x \leq 0 , y \leq 0$ represents the 3rd quadrant

☐ $x \geq 0 , y \leq 0$ represents the 4th quadrant






TRIGONOMETRY


Trig Functions
sin(x)
cos(x)

Lesson (1)

Trigonometric identities

➢ **Trigonometric identity:**

It is an inequality which is true for all values of the variable

Ex) $tan\theta = \frac{sin\theta}{cos\theta}$ is called identity because it is true for all values of θ

➢ **Inequality:** it is not true for all values of the variable

Ex) $sin\theta = \frac{1}{2}$

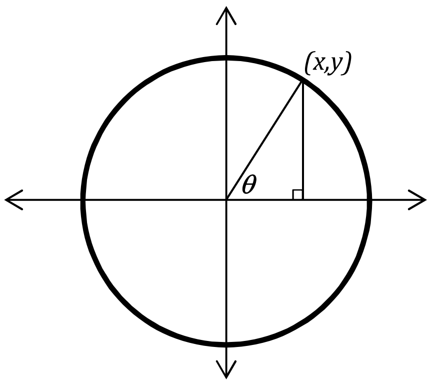
Basic trigonometric identities

(1) $\boldsymbol{tan\theta = \frac{sin\theta}{cos\theta}}$

(2) $\boldsymbol{cot\theta = \frac{cos\theta}{sin\theta}}$

(3) $\boldsymbol{sin\theta = \frac{1}{csc\theta}\ , csc\theta = \frac{1}{sin\theta}}$ *and* $\boldsymbol{cos\theta = \frac{1}{sec\theta}\ ,\ sec\theta = \frac{1}{cos\theta}}$

(4) $\boldsymbol{tan\theta = \frac{1}{cot\theta}\ ,\ cot\theta = \frac{1}{tan\theta}}$



(5) From the unit circle:

$\boldsymbol{x^2 + y^2 = 1}$ then $\boxed{\boldsymbol{sin^2\theta + cos^2\theta = 1}}$

Dividing by $cos^2\theta$

$\frac{sin^2\theta}{cos^2\theta} + \frac{cos^2\theta}{cos^2\theta} = \frac{1}{cos^2\theta}$ then $tan^2\theta + 1 = sec^2\theta$

$$\boxed{sec^2\theta = 1 + tan^2\theta}$$

Dividing by $sin^2\theta$ $\boxed{\csc^2\theta = 1 + \cot^2\theta}$

Sheet (1)

1 Which of the following relations represents an equation and which of them represents an identity :

1) $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

2) $\cot\theta = \frac{-1}{\sqrt{3}}$

3) $\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$

4) $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

5) $\sin^2\theta + \cos^2\theta = 1$

6) $\sin(2\pi - \theta) = -\frac{1}{2}$

....

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....

2 Choose the correct answer from the given ones :

1) $\cos(90° - \theta)\sec(\theta - 90°)$ in the simplest form equals ....

(a) 1 (b) – 1 (c) $\sin^2\theta$ (d) $\cot^2\theta$

2) The expression : $\frac{1 - \cos^2\beta}{\sin^2\beta - 1}$ in the simplest form equals ....

(a) $-\tan^2\beta$ (b) $-\cos^2\beta$ (c) $\tan^2\beta$ (d) $\cot^2\beta$

3) $\frac{1 + \tan^2\theta}{1 + \cot^2\theta}$ in the simplest form equals ....

(a) $\tan^2\theta$ (b) $\cot^2\theta$ (c) 1 (d) $\cos^2\theta$

4) $\sin^2\theta + \cos^2\theta + \tan^2\theta =$ ....

(a) 1 (b) $\cot^2\theta$ (c) $\csc^2\theta$ (d) $\sec^2\theta$

5) $(\tan^2\theta - \sec^2\theta)^5 =$ ....

(a) 1 (b) – 1 (c) 5 (d) – 5

6) $2\sin^2\theta + \cos^2\theta + \frac{1}{\sec^2\theta} =$ ....

(a) 2 (b) 1 (c) $\tan^2\theta$ (d) $\sec^2\theta$

7) $\sin\theta\csc\theta + 2\cos\theta\sec\theta + 3\tan\theta\cot\theta =$ ....

(a) 1 (b) 3 (c) 5 (d) 6

8) In Δ ABC , if $\sin^2 A + \cos^2 B = 1$, then Δ ABC is ....

(a) equilateral. (b) isosceles. (c) scalene. (d) right-angled.

3 **Complete the following "where θ is the measure of an angle in which all trigonometric functions and their reciprocals are defined at it" :**

(1) $\sin\theta \csc\theta =$ ……………

(2) $\cos\theta = \dfrac{1}{\ldots\ldots}$

(3) $\cot\theta \tan\theta =$ ……………

(4) $\dfrac{\sin\theta}{\cos\theta} =$ ……………

(5) $\sin^2\theta + \cos^2\theta =$ ……………

(6) $\sin^2\theta = 1 -$ ……………

(7) $\tan^2\theta + 1 =$ ……………

(8) $\cot^2\theta + 1 =$ ……………

4 **Write in the simplest form each of the following expressions "where θ is the measure of an angle in which all trigonometric functions and their reciprocals are defined at it" :**

1. $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$
2. $\sin\left(\frac{\pi}{2} + \theta\right)\sec(-\theta)$
3. $\cos^2\theta \sec\theta \csc\theta$
4. $\sin\theta\csc\theta - \cos^2\theta$

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5 **Prove the validity of each of the following identities :**

1. $\sin(90^\circ - \mu)\cos\mu = 1 - \sin^2\mu$
2. $\cot^2\mu - \cos^2\mu = \cot^2\mu \cos\mu^2$
3. $\sec^2\beta + \csc^2\beta = \sec^2\beta\csc^2\beta$
4. $\sec\theta - \sin\theta\tan\theta = \cos\theta$

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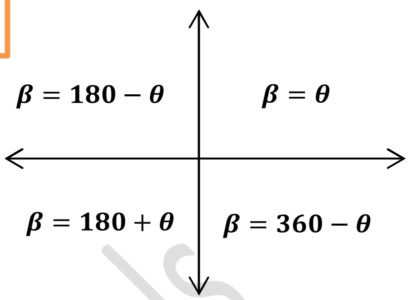
Lesson (2)

Solving trigonometric equations



First: finding the general solution

Steps:

a) Determine the quadrant
b) Find the angle " shift …… "
c) Add $\mathbf{2n\pi}$

- The general solution the equation : cos θ = a is $\theta = \pm\beta + 2\pi n$

- The general solution the equation : sin θ = a is
  $\theta = \beta + 2\pi n$, $\theta = (\pi - \beta) + 2\pi n$

- The general solution the equation : tan θ = a is $\theta = \beta + \pi n$

Sheet (2)

1 Complete the following :

(1) The general solution of the equation : $\sin\theta = 1$ for all values of θ is ……………

(2) The general solution of the equation : $\cos\theta = 1$ for all values of θ is ……………

(3) The general solution of the equation : $\sin\theta = -1$ for all values of θ is ……………

(4) The general solution of the equation : $\cos\theta = -1$ for all values of θ is ……………

(5) The general solution of the equation : $\sin\theta = 0$ for all values of θ is ……………

(6) The general solution of the equation : $\cos\theta = 0$ for all values of θ is ……………

(7) The general solution of the equation : $\tan\theta = 1$ for all values of θ is ……………

(8) The general solution of the equation : $\sin\theta = \cos\theta$ for all values of θ is …………

(9) The solution set of the equation : $\sin\theta = \frac{1}{2}$, where $\theta \in]0 , \frac{\pi}{2}[$ is ……………

2 Choose the correct answer :

(1) If $0° \le \theta < 360°$ and $\sin\theta + 1 = 0$, then $\theta =$ ....

(a) 0° (b) 90° (c) 180° (d) 270°

(2) If $0° \le \theta < 360°$ and $\cos\theta + 1 = 0$, then $\theta =$ ....

(a) 90° (b) 180° (c) 270° (d) 360°

(3) If $0° \le \theta < 360°$ and $\csc\theta - 1 = 0$, then $\theta =$ ....

(a) 0° (b) 90° (c) 180° (d) 270°

(4) The solution set of the equation : $\sqrt{3}\tan\theta = 1$, where $90° < \theta < 270°$ is ....

(a) $\{30°\}$ (b) $\{150°\}$ (c) $\{210°\}$ (d) $\{240°\}$

(5) The solution set of the equation : $\sin\theta + \cos\theta = 0$, where $180° < \theta < 360°$ is ....

(a) $\{210°\}$ (b) $\{225°\}$ (c) $\{240°\}$ (d) $\{315°\}$

(6) If $\theta \in [0 , \pi[$, $\cot\theta = 1$, then $\theta =$ ....

(a) 30° (b) 45° (c) 60° (d) 135°

(7) If $\theta \in [0 , \frac{\pi}{2}[$, $\sin\theta\cot\theta = \frac{1}{2}$, then the solution set is ....

(a) $\varnothing$ (b) $\left\{\frac{\pi}{3}\right\}$ (c) $\left\{\frac{4\pi}{3}\right\}$ (d) $\left\{\frac{5\pi}{3}\right\}$

(8) The solution set of the equation : $\sin^2\theta + 1 = 0$, $\theta \in [0 , \pi[$, is ....

(a) $\{90°\}$ (b) $\{0°\}$ (c) $\{180°\}$ (d) $\varnothing$

(9) If $180° \le \theta < 360°$ and $2\cos\theta + 1 = 0$, then $\theta =$ ....

(a) 210° (b) 240° (c) 300° (d) 330°

(10) The general solution of the equation : $\tan\theta = \frac{1}{\sqrt{3}}$ is .... (where $n \in \mathbb{Z}$)

(a) $\frac{\pi}{6} + n\pi$ (b) $2n\pi \pm \frac{\pi}{6}$ (c) $\frac{\pi}{3} + n\pi$ (d) $2n\pi \pm \frac{\pi}{3}$

(11) The general solution of the equation : $\cos\theta = \frac{1}{2}$ is .... (where $n \in \mathbb{Z}$)

(a) $2n\pi \pm \frac{\pi}{3}$ (b) $2n\pi \pm \frac{\pi}{6}$ (c) $\frac{\pi}{6} + n\pi$ (d) $\frac{\pi}{3} + n\pi$

3 Solve each of the following equations in the interval $\left[0 , \frac{3\pi}{2}\right[$:

(1) $\tan^2 \theta - \tan \theta = 0$

(2) $2 \sin \theta \cos \theta - \cos \theta = 0$

(3) $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$

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4 Find the general solution of each of the following equations :

(1) $\cos \theta = \sin 2\theta$

(2) $\cos 2\theta = \sin \theta$

(3) $\cos 5\theta = \sin 4\theta$

(4) $\sec 4\theta = \csc 2\theta$

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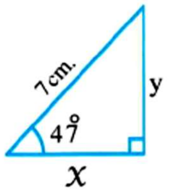
Lesson (3)

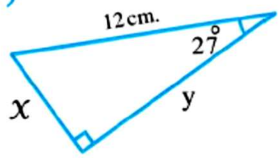
Solving the right-angled triangle

1 Find the value of each of x and y in each of the following figures :

(1)



(2)



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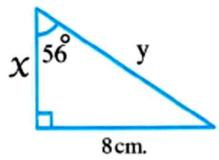

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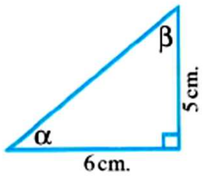
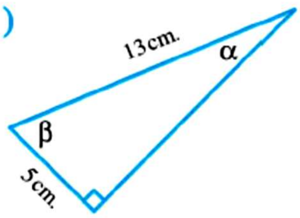
2 Find the value of each of the angles α and β in degree measure in each of the following figures :

(1)



(2)



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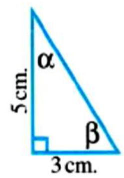

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3 ABC is a right-angled triangle at B. Find AB to one decimal , if :

(1) $m(\angle C) = 32° 18'$ and AC = 25 cm.

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Sheet (3)

1 **ABC is a right-angled triangle at B. Find AB to one decimal , if :**

(1) m (∠ C) = $54° 13'$ and BC = 20 cm. « 27.7 cm. »

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2 **ABC is a right-angled triangle at B. Find m (∠ C) to the nearest minute , if :**

(1) BC = 54 cm. and AC = 88 cm. « $52° 9'$ »

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3 **Solve the triangle ABC which is right-angled at B approximating the measures of angles to the nearest degree and the lengths of sides to the nearest cm. where :**

(1) AB = 4 cm , BC = 6 cm. | (2) AB = 12.5 cm , BC = 17.6 cm.

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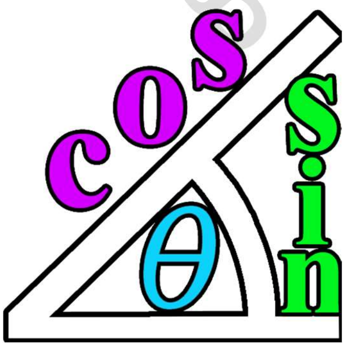


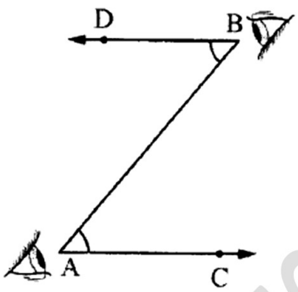
Lesson (4)

Angles of elevation and angles of depression

Angle of elevation

If a person looked from the point A to an object at the point B above his horizontal sight , then the included angle between the horizontal ray $\overrightarrow{AC}$ and the seeing ray to above $\overrightarrow{AB}$ is called the elevation angle of B with respect to A

i.e. $\angle$ CAB is the elevation angle of B with respect to A



Angle of depression

If a person looked from the point B to an object at the point A down his horizontal sight , then the included angle between the horizontal ray $\overrightarrow{BD}$ and the seeing ray to down $\overrightarrow{BA}$ is called the depression angle of A with respect to B

i.e. $\angle$ DBA is the depression angle of A with respect to B

Sheet (4)

1 From a point 8 metres apart from the base of a tree , it was found that the measure of the elevation angle of the top of the tree is 22°
Find the height of the tree to the nearest hundredth. « 3.23 m. »

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2 A man found that the measure of the angle of elevation of the top of a tower , at a distance of 50 m. from its base , is 39° 21̀ Find the height of the tower. « 41 m. »

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3 The length of the thread of a kite is 42 metres. If the measure of the angle which the thread makes with the horizontal ground equals 63° , find to the nearest metre the height of the kite from the surface of the ground. « 37 m. »

………

………

………

………

4 A person observed , from the top of a hill 2.56 km. high , a point on the ground. He found its depression angle measure was 63°. Find the distance between the point and the observer to the nearest metre. « 2873 m. »

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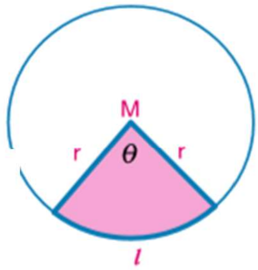
Lesson (5)

The Circular sector



The circular sector: is a part of the surface of the circle bounded by two radii and an arc .

Area of the circular sector $= \frac{1}{2} r^2 \theta^{rad}$ (where θ is the angle of the sector, r is the radius of the circle)

Example

(1) Find the area of the circular sector in which the length of the radius of its circle is 10cm and the measure of its angle is 1.2^{rad}

Solution

Formula: Area of the circular sector $= \frac{1}{2} r^2 \theta^{rad}$

Substituting $r = 10, \theta^{rad} = 1.2^{rad}$: $= \frac{1}{2} (10)^2 \times 1.2 = 60 \text{ cm}^2$

Remember

Relation between the degree measure and the radian measure is:

$$\frac{\theta^{rad}}{\pi} = \frac{x^\circ}{180^\circ}$$

Example

(2) A circular sector in which the length of the radius of its circle equals 16cm, and the measure of its angle equals 120°, find its area to the nearest square centimetre .

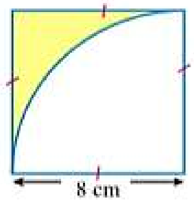
Solution

Formula: area of the sector $= \frac{x^\circ}{360^\circ} \times \pi r^2$

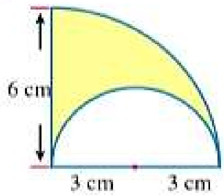
Substituting $r = 16, x^\circ = 120^\circ$: $= \frac{120^\circ}{360^\circ} \times \pi (16)^2 \simeq 268 \text{ cm}^2$

① Find in terms of π the area of the shaded part in each of the following figures:

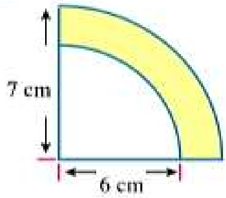
A



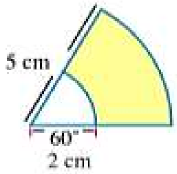
B



C



D



……

……

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……

2 Find to the nearest cm² the area of a circular sector , where the measure of its central angle is 30° and the radius of its circle is of length 3.5 cm. « 3 cm² approximately »

……

……

……

3 Find the area of the circular sector in which the length of the radius of its circle is 10 cm. and the measure of its angle is 1.2^{rad} « 60 cm² »

……

……

……

Sheet (5)

1 **Choose the correct answer from the given ones :**

(1) The area of the circular sector = ……………

(a) $\frac{1}{2}\ell r^2$ (b) $\frac{1}{2} r \theta^{rad}$

(c) the area of the circle $\times \frac{\theta^{rad}}{2\pi}$ (d) the area of the circle $\times \frac{x^\circ}{180^\circ}$

(2) The area of a sector whose arc is of length 10 cm. and the length of the diameter of its circle = 10 cm. equals ……………

(a) 50 cm^2 (b) 25 cm^2 (c) 12.5 cm^2 (d) 100 cm^2

(3) The area of the circular sector in which the measure of its angle is 1.2^{rad} and the length of the radius of its circle is 4 cm. equals ……………

(a) 4.8 cm^2 (b) 9.6 cm^2 (c) 12.8 cm^2 (d) 19.6 cm^2

(4) The perimeter of the circular sector in which the length of its arc is 4 cm. and the length of the diameter of its circle is 10 cm. equals ……………

(a) 14 cm. (b) 20 cm. (c) 30 cm. (d) 40 cm.

(5) The area of the circular sector in which the measure of its angle is 120° , the length of the radius of its circle is 3 cm. equals ……………

(a) 3π cm^2 (b) 6π cm^2 (c) 9π cm^2 (d) 12π cm^2

(6) The area of the circular sector in which , its perimeter is 12 cm. , length of its arc is 6 cm. equals ……………

(a) 6 cm^2 (b) 9 cm^2 (c) 12 cm^2 (d) 18 cm^2

(7) If the perimeter of a sector is 8 cm. and its arc is of length 2 cm. , then its circle is of radius length ……………

(a) 6 cm. (b) 2 cm. (c) 3 cm. (d) 4 cm.

(8) The arc of a sector is of length 3 cm. and the area of this sector is 15 cm^2 , then its circle radius is of length ……………

(a) 5 cm. (b) 10 cm. (c) 2.5 cm. (d) 15 cm.

(9) The perimeter of a sector is 44 cm. Its circle is of radius length 14 cm. , then the length of the arc of the sector = ……………

(a) 16 cm. (b) 8 cm. (c) 32 cm. (d) 4 cm.

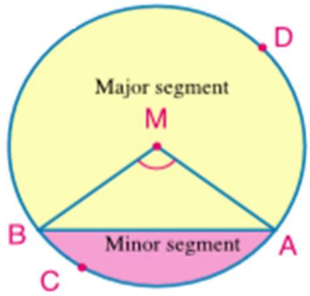
(10) If the area of the circular sector equals 110 cm^2 , the measure of its angle equals 2.2^{rad} , then the length of the radius of its circle equals ……………

(a) 2 cm. (b) 5 cm. (c) 10 cm. (d) 20 cm.

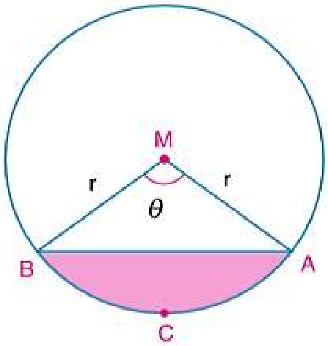
Lesson (6)

Circular Segment

The circular segment is a part of the surface of the circle bounded by an arc and a chord passing by the ends of this arc.



Finding the area of the circular segment:



$$\text{Area of the circular segment} = \frac{1}{2} r^2(\theta^{rad} - \sin \theta)$$

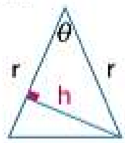
Where r is the length of the radius of its circle , θ is the measure of the angle of the segment.

Remember

Area of the triangle $= \frac{1}{2}$ r× h where:

$\sin \theta = \frac{h}{r}$

$h = r \sin \theta$



Area of the triangle =
$\frac{1}{2} \times r \times r \sin\theta$

Example

(1) Find the area of the circular segment whose length of the radius of its circle equals 8cm, and the measure of its angle equals 150°.

Solution

$\theta^{rad} = 150° \times \frac{\pi}{180°} \simeq \frac{5\pi}{6}$

$\sin\theta = \sin 150°$

Area of the circular segment $= \frac{1}{2} r^2 (\theta^{rad} - \sin\theta)$

Area of the circular segment $= \frac{1}{2} \times 64 \left(\frac{5\pi}{6} - \sin 150°\right) \simeq 67.7758 \text{ cm}^2$

Sheet (6)

1 Complete :

(1) The circular segment is ……………

(2) The area of the circular segment = ……………

(3) The area of the circular segment whose radius length is 10 cm. and its arc is of length 5 cm. is ……………

(4) The area of the circular segment equals the area of the circular sector subtended by the same arc if its central angle is of measure ……………

(5) ABC is a triangle in which : AB = 5 cm. , BC = 8 cm. , m (∠ B) = 60° , then the area of Δ ABC = …………… cm^2

2 Find the area of the circular segment in which :

(1) The length of its chord equals 6 cm. , and the length of the radius of its circle equals 5 cm. « 4 cm^2 approximately »

(2) Its height equals 5 cm. , and the length of the radius of its circle equals 10 cm. « 61 cm^2 approximately »

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…………………………………………………………………………………………………

3

A chord of length 6 cm. is drawn in a circle of radius length 6 cm. Find the area of the minor segment. « 3.26 cm^2 approximately »

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4

The area of a circle is 706.5 cm^2. Find the area of a segment of this circle where the measure of its angle is 135° « 185.52 cm^2 approximately »

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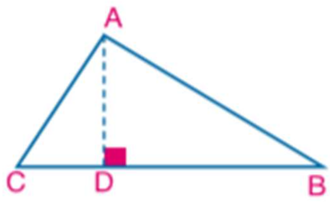
Lesson (7) : Areas

The Area of a triangle in terms of the lengths of two sides and the included angle



From the area of the triangle:

Area of the triangle $= \frac{1}{2}$ BC $\times$ AD

$= \frac{1}{2} \times$ BC $\times$ AB sin B

In general:

Area of the triangle = half the product of the lengths of two sides $\times$ sine the included angle between them.

Example

1) Find the area of the triangle ABC in which AB = 9 cm, AC = 12 cm, m($\angle$A) = 48° approximating the result to the nearest hundredth.

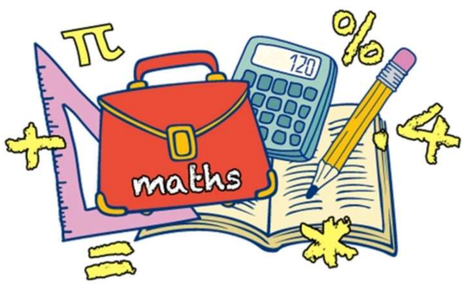
Solution

Area of the triangle A B C $= \frac{1}{2} \times$ A B $\times$ AC sin A

Substituting AB = 9 cm , AC = 12 cm, m($\angle$A) = 48°

Area of the triangle ABC $= \frac{1}{2} \times 9 \times 12 \times \sin 48 \simeq 40.13$ cm^2

→ 1 ÷ 2 × 9 × 1 2 × Sin 4 8 =



Sheet (7)

1 Find the area of the triangle ABC in which : AB = 8 cm. , AC = 10 cm. and m (∠ A) = 48° approximating the result to the nearest hundredth. « 29.73 cm^2 »

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……………………………………………………………………………………

2 The area of the equilateral triangle ABC is $36\sqrt{3}$ cm^2 , then find its side length. « 12 cm. »

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……………………………………………………………………………………

3 Find the area of the quadrilateral in which the lengths of its diagonals are 12 cm. , 16 cm. and the measure of the included angle between them is 68° approximating the result to the nearest square centimetre. « 89 cm^2 »

……………………………………………………………………………………
……………………………………………………………………………………

4 **Find the area of each of the following regular polygons approximating the result to the nearest tenth :**

(1) A regular pentagon of side length equals 16 cm. « 440.4 cm^2 »

(2) A regular hexagon of side length equals 12 cm. « 374.1 cm^2 »

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UnitSummary

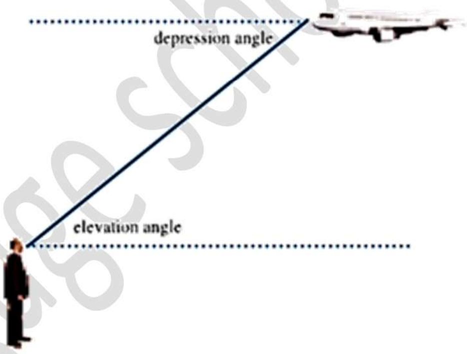
The identity: is true equality for all real values of the variable which each of the two sides of the equality is known.

Pythagorian identites: $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, $1 + \cot^2\theta = \csc^2\theta$

Prove the validity of the identity: to prove the validity of trigonometric identity, we prove that the two functions determining its two sides are equal.

The function: is a true equality for some real numbers which satisfies this equality and is not true for some other which is not satisfy it.

Elevation angle and depression angle:
Elevation or depression angle is the union of the horizontal ray and the initial ray from the body passing through the eye of the observer.
Measure of the elevation angle = measure of the depression angle.
(alternate).



The circular sector: is a part of the surface of the circle bounded by the two radii and an arc .

Area of the circular sector

$= \frac{1}{2} r^2 \theta^{rad}$ (where θ^{rad} is the angle of the sector , **r** is the radius of its circle)

$= \frac{x^\circ}{360^\circ} \times$ Area of the circle (where x° is the degree measure of the angle of the sector)

$= \frac{1}{2} l\, r$ (where l is the length of the arc, r is the radius of its circle)

The circular segment : is a part of the surface of the circle bounded by an arc in it and a chord passes through its ends of this arc.

Area of the segment $= \frac{1}{2} r^2(\theta^{rad} - \sin\theta)$

(where θ is the measure of the central angle of the segment, **r** is the radius of its circle).

Area of the triangle $= \frac{1}{2}$ length of the base $\times$ height

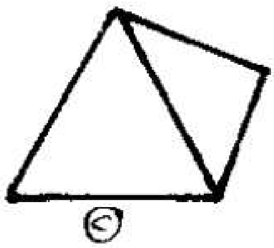
$= \frac{1}{2}$ Product of its sides $\times$ sine the included angle between them.

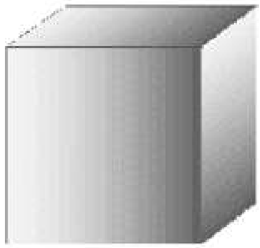
Are of the quadrilateral $= \frac{1}{2}$ product of its diagonals $\times$ sine the included angle between them.

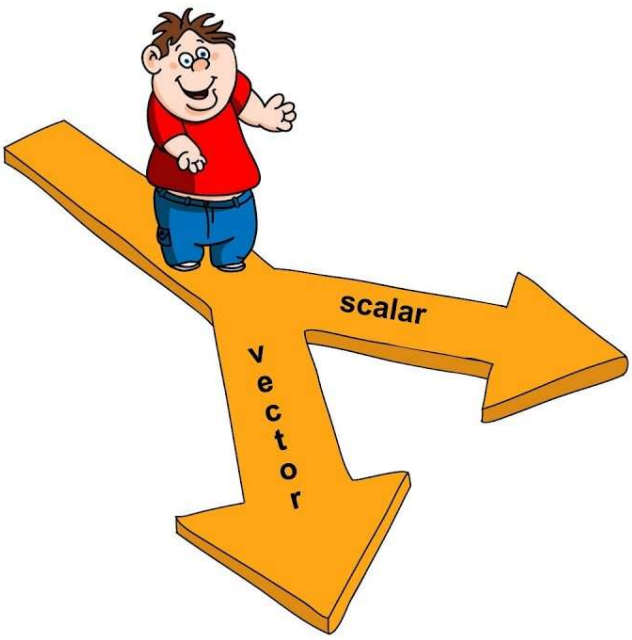
Area of the regular polygon $= \frac{1}{4} n\, x^2 \times \cot\frac{\pi}{n}$

(where n is the number of its sides, x is the length of its side)

Analytic Geometry







Lesson (1)

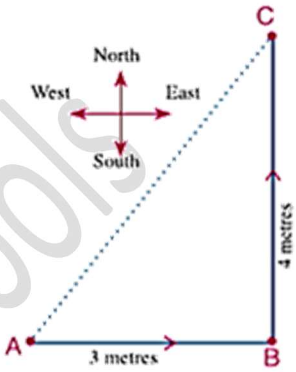
Scalars, Vectors & Directed line segment

Scalar quantities

Scalar quantities are determined completely by their magnitude only such as length, area …

Vector quantities

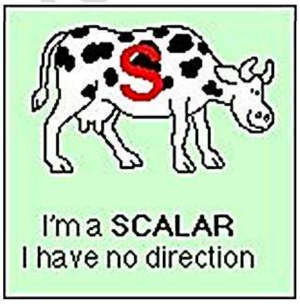
Vector quantities are determined completely by their magnitude and their direction such as velocity, force. …



Notice that:

- **Distance** is a scalar quantity which is the result of AB + BC or CB + BA.
- **Displacement** is the distance between the starting and ending points only and in direction from A to C. i.e to describe the displacement, its magnitude AC and its direction from A to C must be determined.

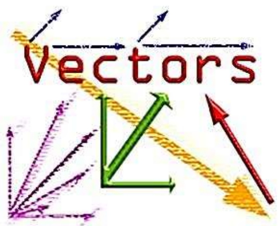
Displacement is a vector quantity which is the distance covered in a certain direction.

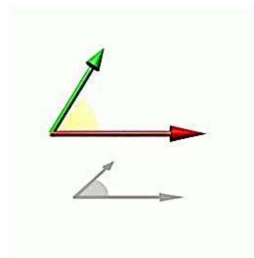


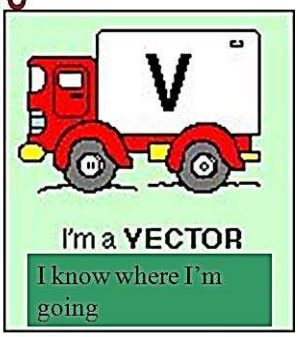
Vector Addition

R = A + B

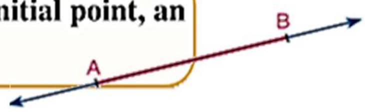
Vectors and Scalars

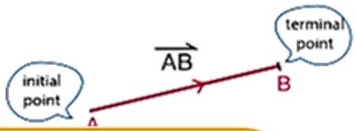






Definition 1 **The directed line segment:** is a line segment which has an initial point, an terminal point and a direction.





Definition 2 **The norm of the directed line segment:** norm of $\overrightarrow{AB}$ is the length of $\overline{AB}$ and is denoted by the symbol $\|\overrightarrow{AB}\|$.

Notice that: $\|\overrightarrow{AB}\| = \|\overrightarrow{BA}\| = AB$

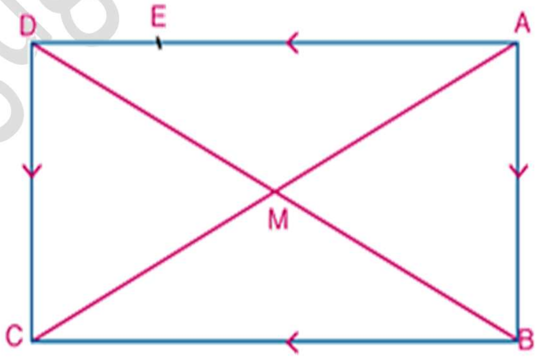
Definition 3 **Equivalent directed line segments:** Two directed line segments are said to be equivalent if they have the same norm and same direction.

Example

1. **In the figure opposite:** ABCD is a rectangle, its diagonals are intersecting at M. $E \in \overline{AD}$ then:

$\overline{AB} \,//\, \overline{CD}$, $AB = CD$, $\overline{BC} \,//\, \overline{AD}$, $BC = AD$ and

$MA = MC = MB = MD$



A $\because \|\overrightarrow{AB}\| = \|\overrightarrow{DC}\|$, $\overrightarrow{AB}$ and $\overrightarrow{DC}$ have the same direction

$\therefore$ $\overrightarrow{AB}$ is equivalent to $\overrightarrow{DC}$

B $\because \|\overrightarrow{AM}\| = \|\overrightarrow{MC}\|$, $\overrightarrow{AM}$ and $\overrightarrow{MC}$ have the same direction

$\therefore$ $\overrightarrow{AM}$ is equivalent to $\overrightarrow{MC}$

C $\because \|\overrightarrow{MA}\| = \|\overrightarrow{MB}\|$, $\overrightarrow{MA}$ and $\overrightarrow{MB}$ have different direction

$\therefore$ $\overrightarrow{MA}$ is not equivalent to $\overrightarrow{MB}$

D $\because \|\overrightarrow{AE}\| \neq \|\overrightarrow{CB}\|$, $\overrightarrow{AE}$ and $\overrightarrow{BC}$ have the same direction

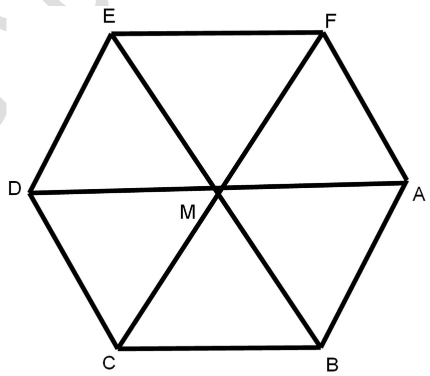
$\therefore$ $\overrightarrow{AE}$ is not equivalent to $\overrightarrow{BC}$

Sheet (1)

1 Complete :

1. to define scalar quantity you should know …….
2. to define vector quantity you should know …….
3. the directed line segment is a line segment which has ………. , …… , …….
4. two directed line segment are equivalent if they have ……
5. in the opposite figure :

ABCDEF is a regular hexagon , then



a) $\overrightarrow{AB}$ is equivalent to ……. And equivalent to …..

b) $\overrightarrow{MD}$ is equivalent to ……. And equivalent to …..

c) $\overrightarrow{MD}$ is equivalent to ……. And equivalent to …..

2 On the lattice , if : A(3,-2) , B(6,2) , C(1,3) , D(4,7)

a) Find : $\|\overrightarrow{AB}\|$ and $\|\overrightarrow{CD}\|$

b) prove that : $\overrightarrow{AB}$ equivalent to $\overrightarrow{CD}$

Lesson (2)

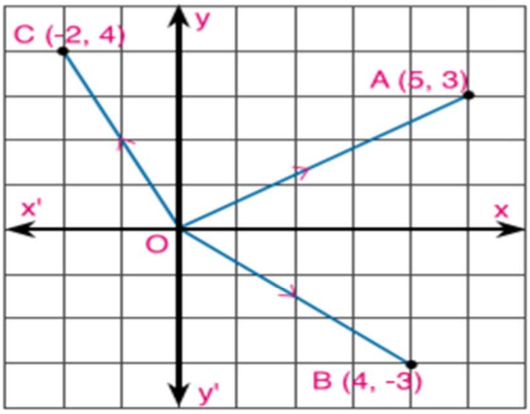
Vectors

Position Vector

Definition 4 **The position vector of a given point with respect to the origin point is the directed line segment which its starting point is the origin point and the given point is its terminal point.**

A (5, 3), B(4, −3), C (−2, 4) then:

- $\overrightarrow{OA}$ is the position vector of the point A with respect to the origin point O, and corresponding to the ordered pair (5, 3) and is written as $\overrightarrow{OA}$ = (5, 3).



Norm of the vector:

Is the length of the line segment representing to the vector.

If: $\overrightarrow{R} = (x, y)$

Then: $\|\overrightarrow{R}\| = \sqrt{x^2 + y^2}$

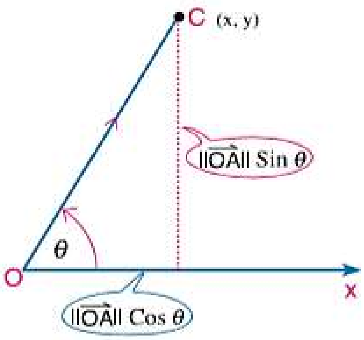
Polar form of position Vector

In the figure opposite: the vector $\overrightarrow{OA}$ makes θ with the positive direction of the x–axis and its norm equals $\|\overrightarrow{OA}\|$. It is possible to express it as follows:

$\overrightarrow{OA} = (\|\overrightarrow{OA}\|, \theta)$ Polar form of the vector.

the coordinates of point A in the orthogonal coordinate plane are:

$$x = \|\overrightarrow{OA}\| \cos\theta \quad , \quad y = \|\overrightarrow{OA}\| \sin\theta \quad , \quad \tan\theta = \frac{y}{x}$$



The unit vector : it is a vector whose norm is unity.

Zero vector : it is a vector whose norm equals zero and denoted by $\vec{O}$ =(0,0)

Parallel and perpendicular vector

For every non zero vectors $\vec{A} = (x_1, y_1)$ and $\vec{B} = (x_2, y_2)$

1) if $\vec{A} \parallel \vec{B}$ Then $\tan\theta_1 = \tan\theta_2$ And $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ And $x_1y_2 - x_2y_1 = 0$	2) if $\vec{A} \perp \vec{B}$ Then $\tan\theta_1 \times \tan\theta_2 = -1$ And $\frac{y_1}{x_1} \times \frac{y_2}{x_2} = -1$ And $x_1x_2 + y_1y_2 = 0$

Example

If $\vec{A} = (6, -8)$, $\vec{B} = (-9, 12)$ and $\vec{C} = (-4, -3)$

(1) Prove that : $\vec{A} \parallel \vec{B}$, $\vec{B} \perp \vec{C}$, $\vec{C} \perp \vec{A}$

……………………………………………………………………………………

……………………………………………………………………………………

……………………………………………………………………………………

Example

If $\vec{M} = (3, 2)$ and $\vec{N} = (2, k)$, **find the value of k in each of the two cases :**

(1) $\vec{M} \parallel \vec{N}$ (2) $\vec{M} \perp \vec{N}$

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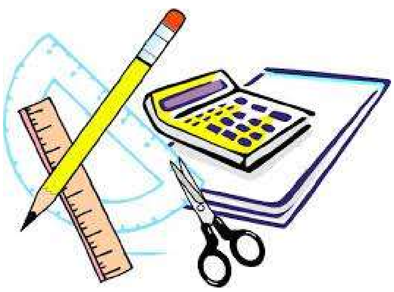

Sheet (2)

1 Complete the following :

(1) The position vector of a given point is ……………

(2) The fundamental unit vector $\overrightarrow{i}$ is the directed line segment to the origin point and its norm is …………… and its direction is ……………

(3) If $\overrightarrow{A} = (4\ ,5)$ and $\overrightarrow{B} = (3\ ,-2)$, then $2\,\overrightarrow{A} + \overrightarrow{B} =$ ……………

(4) If $\overrightarrow{A} = 2\,\overrightarrow{i} + 3\,\overrightarrow{j}$ and $\overrightarrow{B} = 3\,\overrightarrow{i} - \overrightarrow{j}$, then $2\,\overrightarrow{A} - \overrightarrow{B} =$ ……………

(5) If $\overrightarrow{E} = \overrightarrow{O}$ and $\overrightarrow{E} = (2 - a\ , b + 3)$, then a = …………… , b = ……………

(6) If $\overrightarrow{A} = (5\ ,-12)$, then $\|\overrightarrow{A}\| =$ ……………

Choose the correct answer from the given ones :

(1) If $\overrightarrow{A} + \overrightarrow{B} = (8\ ,16)$ and $\overrightarrow{A} = (5\ ,12)$, then $\|\overrightarrow{B}\| =$ ……………

(a) 7 (b) 5 (c) 13 (d) $8\sqrt{5}$

(2) All the following vectors are unit vectors except ……………

(a) (1 , 0) (b) (0.6 , 0.8) (c) (0 , – 1) (d) (1 , 1)

(3) If $\|k\,(3\ ,4)\| = 1$, then k = ……………

(a) $\frac{1}{7}$ (b) $\frac{1}{25}$ (c) $\pm\frac{1}{5}$ (d) ± 5

(4) The vector $\overrightarrow{M} = \left(8\sqrt{2}\ , \frac{\pi}{4}\right)$ is expressed in terms of the fundamental unit vectors by the form ……………

(a) $4\,\overrightarrow{i} + 4\,\overrightarrow{j}$ (b) $8\,\overrightarrow{i} - 8\,\overrightarrow{j}$ (c) $-4\,\overrightarrow{i} - 8\,\overrightarrow{j}$ (d) $8\,\overrightarrow{i} + 8\,\overrightarrow{j}$

(5) If $\overrightarrow{A} = (4\ ,5)$ and $\overrightarrow{B} = (-20\ ,16)$, then the two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ are ……………

(a) perpendicular. (b) parallel. (c) equivalent. (d) otherwise.

(6) If $\overrightarrow{L} = (2\ ,-3)$ and $\overrightarrow{K} = (3\ ,1 - x)$ are parallel , then $x =$ ……………

(a) 4 (b) $\frac{11}{2}$ (c) – 1 (d) – 9

(7) If $\overrightarrow{A} = (x\ ,4)$, $\overrightarrow{B} = (2\ ,y)$ and $\overrightarrow{A} \parallel \overrightarrow{B}$, then ……………

(a) $x + 2y = 0$ (b) $x = 2y$ (c) $xy = 8$ (d) $\frac{\ }{y}\ \cdot 2$

If $\overrightarrow{A} = (6\ ,-8)$, $\overrightarrow{B} = (-9\ ,12)$ and $\overrightarrow{C} = (-4\ ,-3)$

(1) Prove that : $\overrightarrow{A} \parallel \overrightarrow{B}$, $\overrightarrow{B} \perp \overrightarrow{C}$, $\overrightarrow{C} \perp \overrightarrow{A}$

(2) Find : $2\overrightarrow{A} + \overrightarrow{B}$, $\overrightarrow{B} - 2\overrightarrow{C}$, $\frac{1}{2}\overrightarrow{A} + \overrightarrow{B} - 3\overrightarrow{C}$

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If $\overrightarrow{M} = (3\ ,2)$ and $\overrightarrow{N} = (2\ ,k)$, **find the value of k in each of the two cases :**

(1) $\overrightarrow{M} \parallel \overrightarrow{N}$ **(2)** $\overrightarrow{M} \perp \overrightarrow{N}$

....

....

....

If $\| -8\overrightarrow{A} \| = 5 \| k\overrightarrow{A} \|$, **find the value of :** k

....

....

Find the polar form of each of the following vectors :

(1) $\overrightarrow{M} = 8\sqrt{3}\ \overrightarrow{i} + 8\ \overrightarrow{j}$

(2) $\overrightarrow{N} = 3\sqrt{2}\ \overrightarrow{i} + 3\sqrt{2}\ \overrightarrow{j}$

(3) $\overrightarrow{OA} = \left(5\ ,5\sqrt{3}\right)$

(4) $\overrightarrow{B} = \left(7\sqrt{3}\ ,-7\right)$

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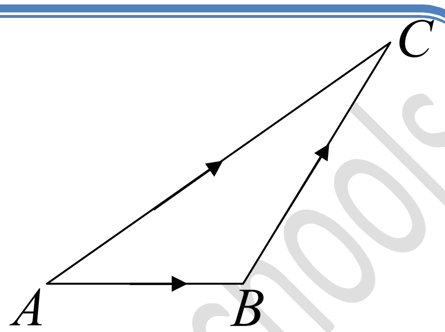
Lesson (3)

Operation On Vectors

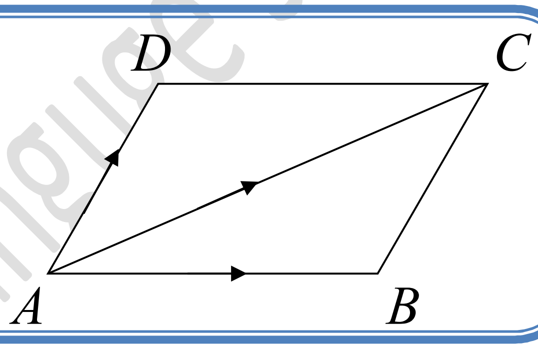
First — Adding vectors geometrically

1] the triangle rule :



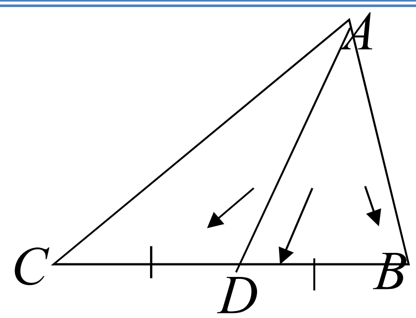
$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

2] the parallelogram rule :



$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AC}$

3] the median rule:

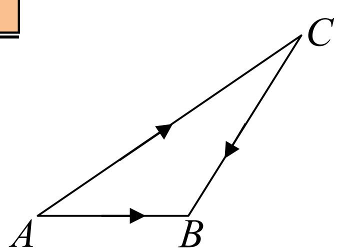


$\overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$

Second — Subtracting two vectors geometrically

$\overrightarrow{AB} - \overrightarrow{AC} = \overrightarrow{CB}$

$\overrightarrow{AB} = \overrightarrow{B} - \overrightarrow{A}$



Example

In the quadrilateral ABCD , prove that :

(1) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$ | (2) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{DC} + \overrightarrow{AD}$

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Sheet (3)

1 Complete :

1. if : $\vec{A} = (-1,5)$, $\vec{B} = (2,1)$,then $\|\overrightarrow{AB}\|$ = ……
2. if : $\vec{A} = (4,-2)$, $\overrightarrow{AB} = (3,5)$, then $\vec{B}$ = …..
3. if : M is a midpoint of $\overline{XY}$,then $\overrightarrow{XM} + \overrightarrow{YM}$ = ……
4. if : ABC is a triangle ,then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$ = ….
5. if : ABC is a triangle ,then $\overrightarrow{AB} - \overrightarrow{CB}$ = ……. , $\overrightarrow{BA} - \overrightarrow{BC}$ = …….

2 **ABCD is a trapezium in which in which $\overline{AD} // \overline{BC}$, E is the midpoint of $\overline{AB}$**

F is the midpoint of $\overline{DC}$.

prove that : $\overrightarrow{AD} + \overrightarrow{BC} = 2\,\overrightarrow{EF}$

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3 **ABCD is a quadrilateral in which : $\overrightarrow{BC} = 3\,\overrightarrow{AD}$ .prove that :**

a) ABCD is a trapezium

b) $\overrightarrow{AC} + \overrightarrow{BD} = 4\,\overrightarrow{EF}$

…………………………………………………………………………………………

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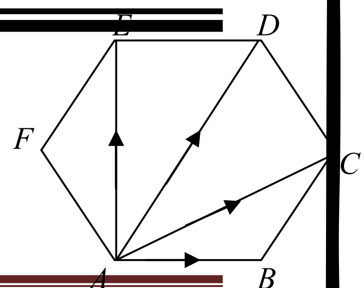
4 **ABCDEF is regular hexagon prove that :**

$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AE} + \overrightarrow{AF} = 2\,\overrightarrow{AD}$



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Lesson (4)

Application on Vectors

First Geometric applications

We know that if $\overrightarrow{AB} = k\,\overrightarrow{DC}$, $k \neq 0$, then $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are :

- carried by the same straight line

i.e. : A , B , C , D are collinear.

or

- carried by two parallel straight lines

i.e. : $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$

Remark

If ABCD is a quadrilateral in which $\overrightarrow{AB} = k\,\overrightarrow{DC}$, $k \neq 0$, then $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, $\|\overrightarrow{AB}\| = |k|\,\|\overrightarrow{DC}\|$ and vise versa.

Example

Use vectors to prove that : the points A (1, 4), B(–1, –2), C(2, –3) are vertices of right angled triangle at B.

……………………………………………………………………………………

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Example

Use the vectors to prove that: the points A (3, 4), B(1, –1), C(–4, –3), D(2, 2) are vertices of a rhombus.

……………………………………………………………………………………

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Second Physical applications

1 The resultant force

- **The force :** is a vector passes through a given point and acts along a straight line.
- **The force :** is represented by a directed line segment and it is drawn by a suitable drawing scale.

For example :

1. A force of magnitude $F_1 = 10$ Newton acts in the East direction.

$\overrightarrow{F_1} = 10\,\vec{e}$

$\overrightarrow{F_1}$ is represented by a directed line segment of length 2 cm.

Remember that :

1. Consider $\vec{e}$ a unit vector in the East direction.
2. Choose a suitable drawing scale "Each 5 Newton is represented on drawing by 1 cm".

Example

If the forces: $\overrightarrow{F_1} = 2\,\vec{i} + \vec{j}$, $\overrightarrow{F_2} = \vec{i} + 7\,\vec{j}$, $\overrightarrow{F_3} = \vec{i} - 5\,\vec{j}$ act on a particle, Calculate the magnitude and direction of their resultant (forces are measured in Newton).

....

....

Relative Velocity

$$\overrightarrow{V_{BA}} = \overrightarrow{V_B} - \overrightarrow{V_A}$$

Example

A car (A) moves on a straight road with speed 70 km/h, A car (B) moves on the same road with speed 90 km/h. Find the relative velocity of car (A) with respect to car (B) when:

A. The two cars move in the same direction.

B. The two cars move in the opposite direction.

....

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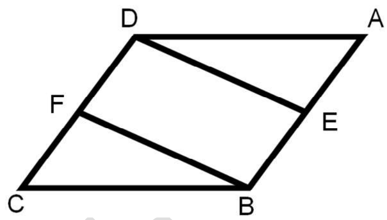
Sheet (4)

First Geometry

1 ABCD **is a parallelogram** ,E **is a midpoint of** AB

F **is a midpoint of** DC

Prove that : DEBF **is a parallelogram**



2 ABCD **is a quadrilateral** , **if** $\overrightarrow{AC} + \overrightarrow{BD} = 2\,\overrightarrow{DC}$ **prove that :**

ABCD **is a parallelogram**

3 **using vectors prove that** : A(3,4) , B(1,-1) , C(-4,-3) , D(-2,2)

are vertices of a rhombus

4 **using vectors prove that** : A(1,3) , B(6, 1) , C(4,-4) , D(-1,-2)

are vertices of a square and find its area.

5 **ABCD is a trapezium** , AD//BC

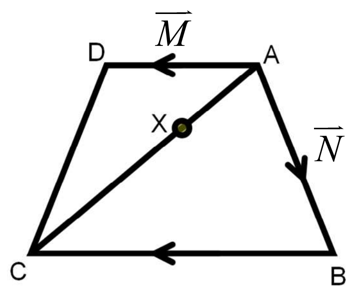
AD $= \frac{1}{2}$ BC , $\overrightarrow{AB} = \overrightarrow{N}$, $\overrightarrow{AD} = \overrightarrow{M}$

a) **Express in term of** $\overrightarrow{M}$ **and** $\overrightarrow{N}$ **each of the following** :

$\overrightarrow{BC}$, $\overrightarrow{AC}$, $\overrightarrow{DC}$, $\overrightarrow{DB}$

b)**if** : X ∈ $\overrightarrow{AC}$ **where** AX $= \frac{1}{3} \times$AC

prove that : the point D , X **and** B **are collinear** .



Second Physical application

1 Complete:

1. If: $\overrightarrow{F_1}$ = i - 3j , $\overrightarrow{F_2}$ = 3i – j act on a particle, then the norm of the resultant = ….N

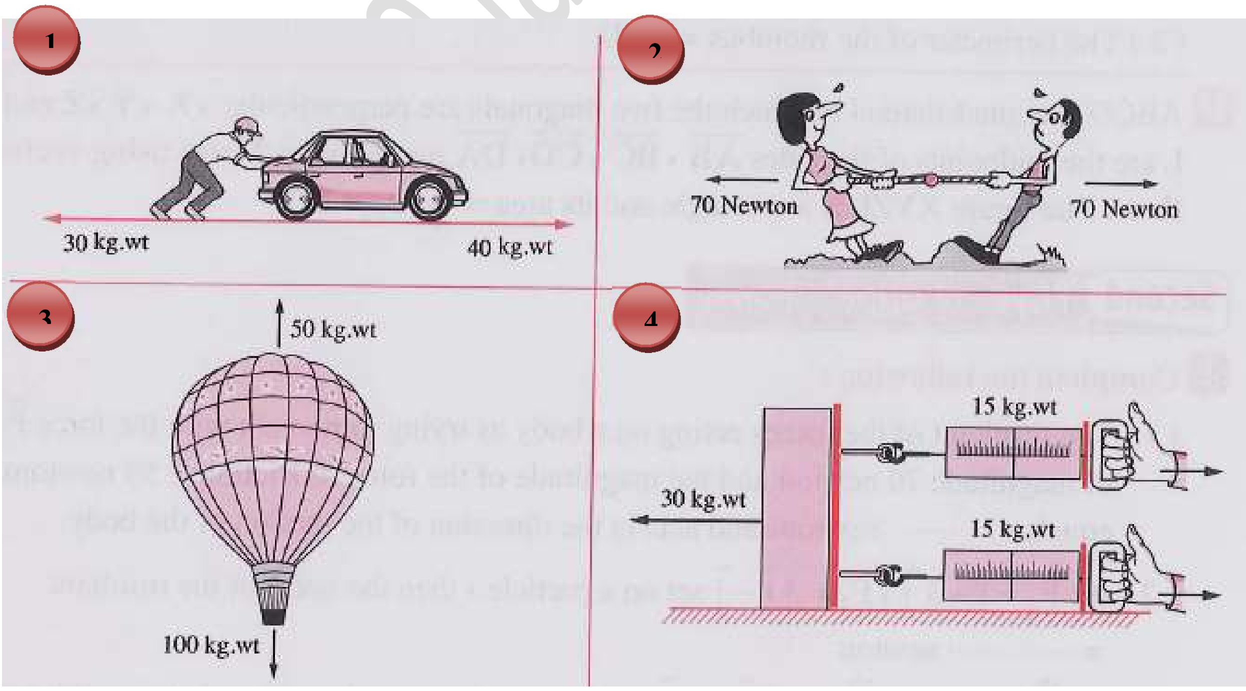
2. If: $\overrightarrow{F_1}$ =(a,b) , $\overrightarrow{F_2}$ = -3i + 4j act on a particle and the system is in equilibrium ,then a = …… , b = …..

3. If: $\overrightarrow{V_A}$ = 12 $\vec{e}$, $\overrightarrow{V_B}$ = 8 $\vec{e}$, then $\vec{V}_{AB}$ = ……

4. If : : $\overrightarrow{V_A}$ = 120 $\vec{e}$, $\overrightarrow{V_B}$ = -80 $\vec{e}$, then $\vec{V}_{BA}$ = …… , $\vec{V}_{AB}$ = ……

5. If: $\vec{V}_{AB}$ = 75 $\vec{e}$, $\overrightarrow{V_A}$ = -60 $\vec{e}$, then $\vec{V}_{BA}$ = …… , $\overrightarrow{V_B}$ = ……

2 Find the resultant force $\vec{F}$ acting in each of the following:



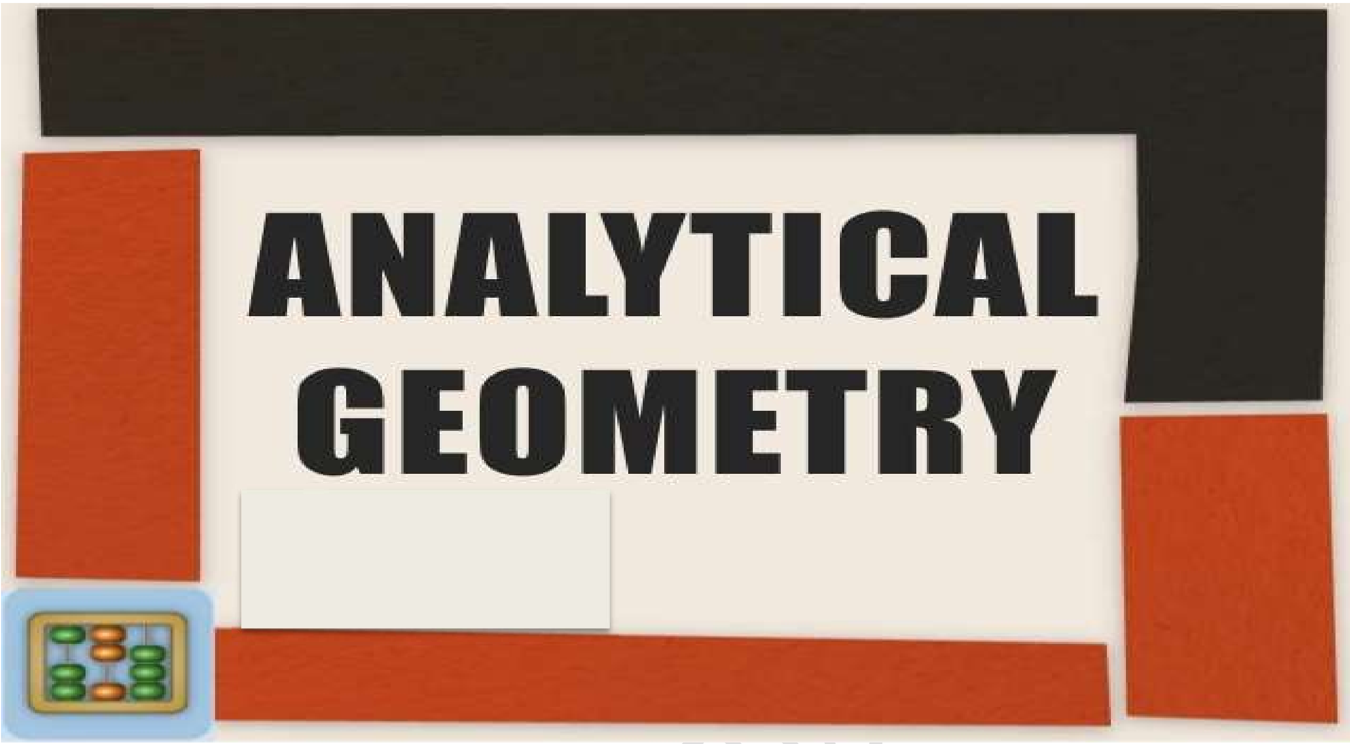
3 In each of the following, the two forces $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$ act at a particle. Show the magnitude and the direction of the resultant of each two forces:

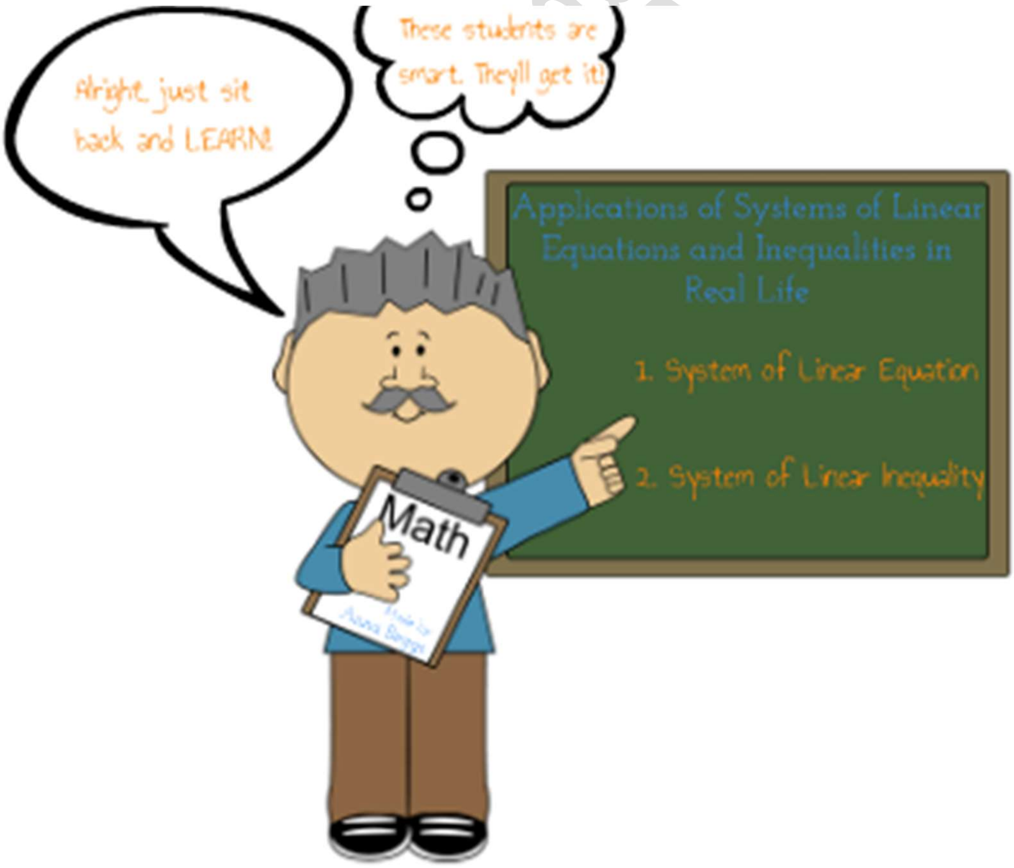
1. $F_1 = 15$ newtons acts in the east direction,
   $F_2 = 40$ newtons acts in the west direction.
2. $F_1 = 34$ gm.wt. acts in the north east direction,
   $F_2 = 34$ gm.wt. acts in the south west direction.
3. $F_1 = 50$ dyne acts in $60°$ west of the north direction,
   $F_2 = 50$ dyne acts in $30°$ south of the east direction.
4. $F_1 = 30$ newtons acts in $20°$ east of the north direction ,
   $F_2 = 30$ newtons acts in $70°$ north of the east direction.

4 Forces $\overrightarrow{F_1} = 7i - 5j$, $\overrightarrow{F_2} = ai + 3j$, $\overrightarrow{F_3} = -4i + (b-3)j$,find the values of a and b if:

(1) The system of forces are in equilibrium.

(2) The resultant of the forces = -5j


ANALYTICAL
GEOMETRY


Alright, just sit back and LEARN!
These students are smart. They'll get it!
Applications of Systems of Linear Equations and Inequalities in Real Life
1. System of Linear Equation
2. System of Linear Inequality
Math

Lesson (1)

Division of a line segment

First: Finding the Coordinates of the point of division of a line segment by a certain ratio:

1- Internal division

If $C \in \overrightarrow{AB}$, then point C divides $\overrightarrow{AB}$ internally by the ratio $m_2 : m_1$ where $\frac{m_2}{m_1} > 0$ then $\frac{AC}{CB} = \frac{m_2}{m_1}$ and for the two directed segments $\overrightarrow{AC}$, $\overrightarrow{CB}$ The same direction i.e.: $m_1 \times \overrightarrow{AC} = m_2 \times \overrightarrow{CB}$

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x, y)$

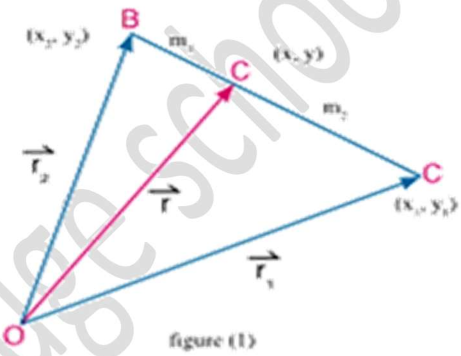


figure (1)

Then $\overrightarrow{r}\,(m_1 + m_2) = m_1 \overrightarrow{r_1} + m_2 \overrightarrow{r_2}$

i.e.: $\boxed{\overrightarrow{r} = \frac{m_1 \overrightarrow{r_1} + m_2 \overrightarrow{r_2}}{m_1 + m_2}}$ **which is called the vector form**

Example

1. If A (2, − 1), B (− 3, 4), find the coordinates of point C which divides $\overrightarrow{AB}$ internally by the ratio 3 : 2 in the vector form.

Solution

Let C (x, y)

$\because A(2, -1) \quad \therefore \overrightarrow{r_1} = (2, -1) \quad , \quad \because B(-3, 4) \quad \therefore \overrightarrow{r_2} = (-3, 4)$

$m_2 : m_1 = 3 : 2$

$\because \overrightarrow{r} = \frac{m_1 \overrightarrow{r_1} + m_2 \overrightarrow{r_2}}{m_1 + m_2}$

$\therefore \overrightarrow{r} = \frac{2(2, -1) + 3(-3, 4)}{2 + 3} = \frac{(4, -2) + (-9, 12)}{5} = \frac{(-5, 10)}{5} = (-1, 2)$

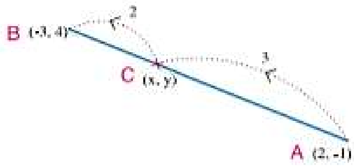
∴ The coordinates of point C are (−1, 2)

Cartesian form:

$$(x, y) = \frac{m_1(x_1, y_1) + m_2(x_2, y_2)}{m_1 + m_2} = \frac{(m_1 x_1 + m_2 x_2,\ m_1 y_1 + m_2 y_2)}{m_1 + m_2}$$

From that we get: $\boxed{(x\ ,\ y) = \left(\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}\ ,\ \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}\right)}$

Example



2 Solve the previous example using the Cartesian form.

Solution

$$(x, y) = \left(\frac{2 \times 2 + 3 \times -3}{2 + 3}\ ,\ \frac{2 \times -1 + 3 \times 4}{2 + 3}\right) = (-1, 2)$$

2- External diviaion

If $C \in \overleftrightarrow{AB}$, $C \notin \overline{BA}$, then C divides $\overrightarrow{AB}$ externally by the ratio $m_2 : m_1$ where $\frac{m_2}{m_1} < 0$ then one of the two values m_1 or m_2 is positive and the other is negative, then the following figure illustrates that there are two probabilities:

$C \in \overrightarrow{AB}$, $C \notin \overline{AB}$

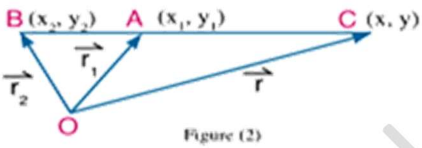


Figure (2)

$C \in \overrightarrow{AB}$, $C \notin \overline{AB}$

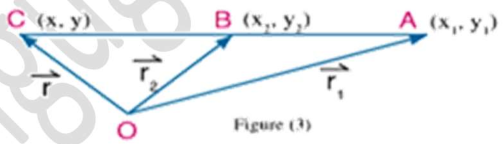
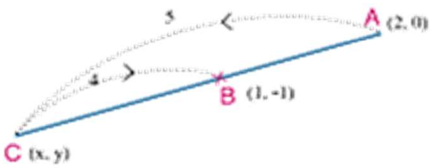


Figure (3)

Example

3 If A (2, 0), B (1, −1), find the coordinates of point C which divides $\overrightarrow{AB}$ externally by the ratio 5 : 4.

Solution



$\because \overrightarrow{r_1} = (2, 0),\ \overrightarrow{r_2} = (1, -1)$

$, m_2 : m_1 = 5 : -4 \quad \therefore \frac{m_2}{m_1} < 0$ negative — by substituting

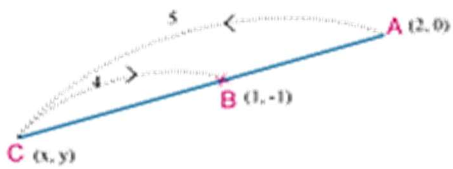
$, \overrightarrow{r} = \frac{m_1 \overrightarrow{r_1} + m_2 \overrightarrow{r_2}}{m_1 + m_2}$ — mathematical formula for the rule

$\therefore \overrightarrow{r} = \frac{-4(2, 0) + 5(1, -1)}{-4 + 5}$ — by distributing

$\overrightarrow{r} = (-8 + 5, 0 - 5) = (-3, -5)$ — by adding and simplifying

$\therefore$ The coordinates of point C are (−3, −5)

Cartesian form:



$$(x, y) = \left(\frac{-4 \times 2 + 5 \times 1}{-4 + 5}, \frac{-4 \times 0 + 5 \times -1}{-4 + 5}\right)$$

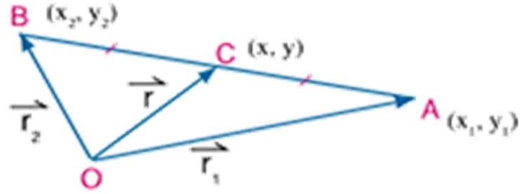
$$= (-3, -5)$$

Notice that:

If C is the midpoint of $\overrightarrow{AB}$ where A (x_1, y_1), B(x_2, y_2)
then: $m_1 = m_2 = m$ then

$$\vec{r} = \frac{\vec{r_1} + \vec{r_2}}{2} \qquad \textbf{Vector form}$$

$$(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) \qquad \textbf{Cartesian form}$$



Second : Finding the ratio of Division

If point C divides $\overrightarrow{AB}$ by the ratio $m_2 : m_1$ and:

1- The ratio of division $\frac{m_2}{m_1} > 0$ then the division is internal.

2- The ratio of division $\frac{m_2}{m_1} < 0$ then the division is external.

Example

4 If A (5, 2), B (2, − 1) , find the ratio by which $\overrightarrow{AB}$ is divided by the points of intersection of $\overleftrightarrow{AB}$ with the two axes, showing the type of division in each case, then find the coordinates of the division point.

Solution

First: let the x-axis intersects $\overrightarrow{AB}$ at point C (x, 0)

where $\frac{AC}{CB} = \frac{m_2}{m_1}$ then: $y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$

$\therefore 0 = \frac{m_1 (2) + m_2 (-1)}{m_1 + m_2} = \frac{2m_1 - m_2}{m_1 + m_2}$

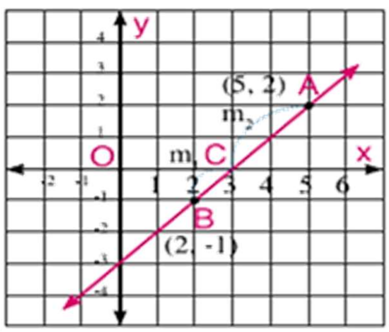
$\therefore 2m_1 = m_2$ $\qquad \therefore \frac{m_2}{m_1} = \frac{2}{1}$ (ratio of division)

$\because \frac{m_2}{m_1} > 0$

$\therefore$ The division is internal by the ratio 2 : 1

$\therefore$ The coordinates are $C\left(\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, 0\right) = \left(\frac{1 \times 5 + 2 \times 2}{1 + 2}, 0\right)$

$= (3, 0)$



Second: The straight line intersects the y-axis at point D

Let the coordinates of D be (0, y)

where $\frac{AD}{DB} = \frac{m_2}{m_1}$ then $x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

$\therefore 0 = \frac{m_1 \times 5 + m_2 \times 2}{m_1 + m_2}$

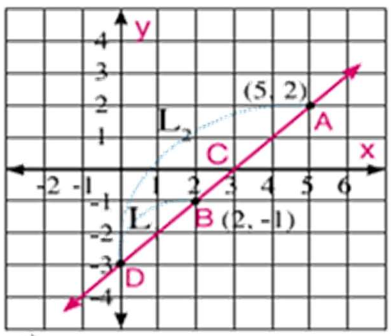
$\therefore 2\, m_2 = -5m_1$ $\qquad \therefore \frac{m_2}{m_1} = -\frac{5}{2}$ (ratio of division)

$\because \frac{m_2}{m_1} < 0$

$\therefore$ The division is external by the ratio 5 : 2

The coordinates of the point D are $(0, y) = \left(0\ ,\ \frac{-2 \times 2 + 5 \times -1}{-2 + 5}\right)$

$\therefore (0, -3)$



Sheet (1)

1 Complete the following :

(1) If A = (3 , 6) , B = (– 7 , 4) , then the midpoint of $\overline{AB}$ = (.............. , ...)

(2) If M is the point of intersection of the two diagonals of the parallelogram ABCD where A = (3 , 7) , C = (– 3 , 1) , then M = (.............. , ...)

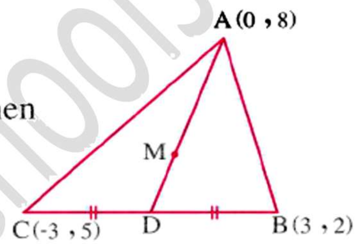
(3) If the point (3 , 6) is the midpoint of $\overline{AB}$ where A = (– 3 , 7) , then the point B = (.............. , ...)

(4) **In the opposite figure :**

$\overline{AD}$ is a median in Δ ABC , M is the point of intersection of its medians where A = (0 , 8) , B = (3 , 2) , C = (– 3 , 5) , then

the point D = (.............. , ...)

the point M = (.............. , ...)



2 If A = (– 3 , – 7) , B = (4 , 0) , find the coordinates of the point C which divides $\overrightarrow{AB}$ by the ratio 5 : 2 internally.

« (2 , – 2) »

....

....

3 If A = (0 , – 3) , B = (3 , 6) , find the coordinates of the point C which divides $\overrightarrow{BA}$ internally by the ratio 1 : 2

« (2 , 3) »

....

....

4 If A = (4 , 3) , B = (– 3 , 5) , find the point C $\in \overleftrightarrow{AB}$ where 3 AC = 5 CB

....

....

....

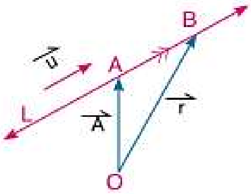
....

Lesson (2)

Equation of straight line

Equaltion of the straight line given a point belonging to it and a direction vector to it

First: Vector form $\boxed{\overrightarrow{r} = \overrightarrow{A} + K\,\overrightarrow{u}}$



Example

(1) Write the vector equation of the straight line which passes through point (2, –3) and its direction vector is (1, 2).

Solution

Let the straight line pass through point A (2, –3) and $\overrightarrow{u} = (1, 2)$

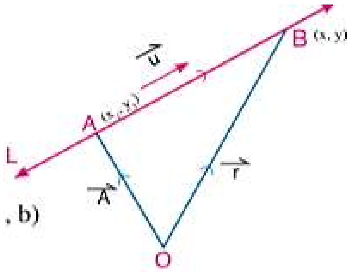
$\because\ \overrightarrow{r} = \overrightarrow{A} + K\,\overrightarrow{u}$ vector form of the equation of the straight line.

$\therefore$ The vector equation of the straight line is $\overrightarrow{r} = (2, -3) + K(1, 2)$.

Second: The parametric equations

The vector equation is $\overrightarrow{r} = \overrightarrow{A} + K\,\overrightarrow{u}$

$\boxed{x = x_1 + k\,a \quad , \quad y = y_1 + kb}$



Third : Cartesian Equation

Eliminating K from the parametric equations : $x = x_1 + ka$, $y = y_1 + kb$

We get the equation: $\frac{x - x_1}{a} = \frac{y - y_1}{b}$ i.e.: $\frac{b}{a} = \frac{y - y_1}{x - x_1}$

Put $\frac{b}{a} = m$ (where m in the slope of the line), then the equation becomes in the form: $m = \frac{y - y_1}{x - x_1}$

Example

(3) Find the Cartesian equation of the straight line which passes through the point (3,–4) and its direction vector is (2, –1)

Solution

$m = \frac{-1}{2}$ Slope of the line $m = \frac{b}{a}$

$m = \frac{y - y_1}{x - x_1}$ equation of the line given its slope and a point belonging to it.

$\frac{-1}{2} = \frac{y - (-4)}{x - 3}$ $m = \frac{-1}{2}$, $x_1 = 3$, $y_1 = -4$

$2y + 8 = -x + 3$ Product of means = product of extrems.

$x + 2y + 5 = 0$ general form.

Sheet (2)

Find the equation of the S.t line

1 Passing through (1 , 3) and its slope = $\frac{-2}{3}$

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2 Passing through the point (3 , -2) and its slope is -2

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3 Passing through the two points (3 , 1) and (5 , 4)

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4 Passing through the point (0 , -5) and makes with the positive direction of X – axis an angle of measure 135^0.

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5 Passing through the point (-2 , 1) and parallel to the straight line

$\vec{r} = (2,-3) + k(1,0)$

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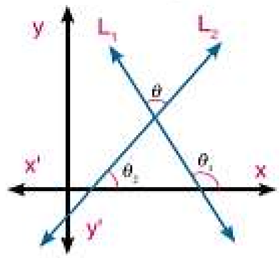
Lesson (3)

The angle between two

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \text{ where } m_1 m_2 \neq -1$$



1. Find the measure of the acute angle between the two straight lines whose equations are
$3x - 4y - 11 = 0$, $x + 7y + 5 = 0$

Solution

A We find the slope of each straight line:

$m_1 = \frac{-3}{-4} = \frac{3}{4}$ slope of the first line

$m_2 = \frac{-1}{7}$ slope of the second line

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ Formula

Remember
Slope of the straight line whose equation $ax + by + c = 0$ equals $\frac{-a}{b}$

$\tan\theta = \left|\frac{\frac{3}{4} - (-\frac{1}{7})}{1 + \frac{3}{4}(-\frac{1}{7})}\right|$ substituting the values of m_1, m_2

$= \left|\frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{28}}\right| = \left|\frac{\frac{21+4}{28}}{\frac{28-3}{28}}\right| = 1$

$\theta = 45^\circ$

Sheet (3)

1 Find the measure of the acute angle between the two straight lines whose slopes are :

(1) $\frac{-3}{4}$, -7 (2) $\frac{1}{2}$, $\frac{2}{9}$ (3) $\frac{3}{4}$, $-\frac{2}{3}$

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2 Find the measure of the acute angle between each of the following pairs of straight lines :

(1) $L_1 : \vec{r} = (0 , -2) + k (3 , -1)$, $L_2 : \vec{r} = (0 , 5) + k (2 , 1)$

(2) $L_1 : \vec{r} = k (1 , 0)$, $L_2 : \vec{r} = (3 , -2) + k (1 , -2)$

(3) $L_1 : \vec{r} = (0 , 1) + k (1 , 1)$, $L_2 : 2x - y - 3 = 0$

(4) $L_1 : 2x + 3y = 15$, $L_2 : \vec{r} = (-2 , -1) + k (1 , -3)$

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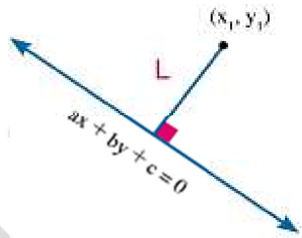
Lesson (4)

The length of the perpendicular from a point to a line

Finding the length of the perpendicular from a point to a straight line

$$L = \frac{|a x_1 + b y_1 + c|}{\sqrt{a^2 + b^2}}$$



Example

1. Find the length of the perpendicular from the point (4, –5) to the straight line $\vec{r} = (0, 2) + K(4, 3)$.

Solution

Let $(x, y) = (0, 2) + K\ (4, 3)$

$\therefore\ x = 4K \ ,\quad y = 2 + 3K$ (parametric equations to the vector equation)

$\frac{x}{4} = \frac{y - 2}{3}$ by eliminating K

$3x = 4y - 8$ Product of means = product of exterems

$3x - 4y + 8 = 0$ Cartesian equation

$L = \frac{|a x_1 + b y_1 + c|}{\sqrt{a^2 + b^2}}$ Formula

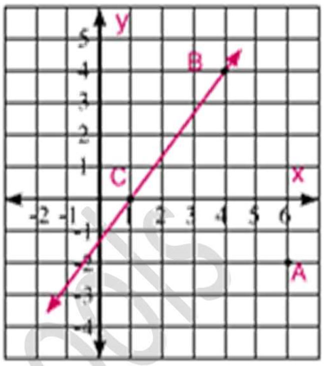
Substituting: $a = 3\ ,\ b = -4\ ,\ c = 8\ ,\ x_1 = 4\ ,\ y_1 = -5$

$$L = \frac{|3 \times 4 - 4 \times -5 + 8|}{\sqrt{3^2 + 4^2}}$$

$$= \frac{|12 + 20 + 8|}{\sqrt{9 + 16}} = \frac{|40|}{\sqrt{25}} = \frac{40}{5} = 8 \text{ unit of length}$$

Example

2 **In the figure opposite**: Find the length of the perpendicular drawn from the point A (6, −2) to the straight line passing through the points B (4, 4), C (1, 0), then find the area of the triangle ABC.



Solution

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Formula

$\because C(1, 0)\ ,\ B(4, 4)$

$\therefore m = \frac{4-0}{4-1} = \frac{4}{3}$ Substituting the point (4, 4), (1, 0)

$m = \frac{y - y_1}{x - x_1}$ equation of the line given the slope and a point belonging to it

$\frac{4}{3} = \frac{y-0}{x-1}$ substituting $m = \frac{4}{3}$

Then: $4x - 3y - 4 = 0$ Cartesian equation

$L = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ formula

length of the perpendicular from the point A (6, −2) to the line : $4x - 3y - 4 = 0$

is: $L = \frac{|4 \times 6 - 3 \times -2 - 4|}{\sqrt{4^2 + 3^2}} = \frac{|24 + 6 - 4|}{\sqrt{25}} = \frac{26}{5} = 5\frac{1}{5}$ unit of length

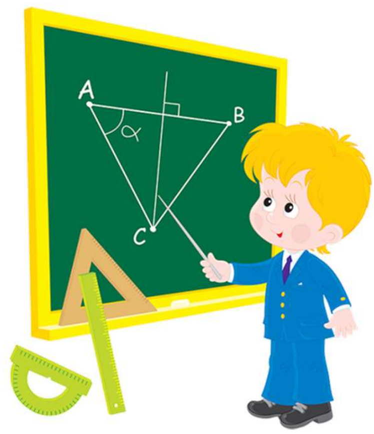
Consider $\overline{BC}$ is the base of the triangle ABC

$\because BC = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ formula

$= \sqrt{(4-1)^2 + (4-0)^2} = 5$ units substituting the points (4, 4), (1, 0)

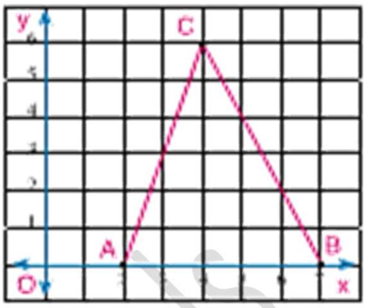
Area of the triangle ABC $= \frac{1}{2}$ length of base × height formula

$= \frac{1}{2} \times 5 \times \frac{26}{5} = 13$ square unit



Sheet (4)

First : Complete each of the following:



(1) The figure opposite shows karim's house A (2, 0) and the school B (7, 0) and the mosque C (4, 6): Complete each of the following:

- (A) The equation of $\overleftrightarrow{AB}$ is
- (B) The length of $\overline{AB}$ equals
- (C) Shortest distacnce between the Mosque C and the road from the house to the school equals
- (D) Measure of the acute angle between the straight lines $\overleftrightarrow{AC}$ and $Y = 0$ equals
- (E) Area of ($\triangle$ ABC) equals

Second : Multiple choice

(2) Length of perpendicular from the point $(-3 , 5)$ on the y-axis equals

(A) 2 (B) 3 (C) 5 (D) 8

(3) The distance between the straight lines whose equations $y - 3 = 0$, $y + 2 = 0$ equals

(A) 1 (B) 2 (C) 3 (D) 5

(4) Length of perpendicular from the point $(1 , 1)$ to the straight line whose equation $x + y = 0$ equals

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) $2\sqrt{2}$

(5) If the length of perpendicular drawn fron $(3 , 1)$ to the straight line whose equation $3x - 4y + c = 0$ equals 2 unit of length, then C equals

(A) Zero (B) 3 (C) 5 (D) 7

(6) Find the length of the perpendicular drawn from (A) to the straight line L in exercises (A) - (D)

- (A) $A(0 , 0)$, $L : \vec{r} = (0 , 5) + t(3 , 4)$
- (B) $A(2 , -4)$, $L : 12x + 5y - 43 = 0$
- (C) $A(5 , 2)$, $L : 8x + 15y - 19 = 0$
- (D) $A(-2 , -1)$, $L : \vec{r} = (0 , -7) + t(1 , 2)$

Lesson (5)

General equation of st.line passing through the point of the intersection of two lines

General equation of the straight line passing through the point of intersection of two given lines

$$a_1 x + b_1 y + c_1 + k(a_2 x + b_2 y + c_2) = 0$$

Example

1. Find the equation of the straight line passing through the point A (– 2, 4) and the point of intersection of the two lines:

$x + 2y - 5 = 0$, $2x - 3y + 4 = 0$

Solution

$a_1 x + b_1 y + c + k(a_2 x + b_2 y + c) = 0$	general equation
$x + 2y - 5 + k(2x - 3y + 4) = 0$	substituting the two equations
$-2 + 2 \times 4 - 5 + k(2\times -2 - 3\times 4 + 4) = 0$	substituting $x = -2$, $y = 4$
$1 - 12k = 0$ i.e. $k = \frac{1}{12}$	Simplify
$x + 2y - 5 + \frac{1}{12}(2x - 3y + 4) = 0$	Substituting the value of k
$12x + 24y - 60 + 2x - 3y + 4 = 0$	multiply both sides by 12
$14x + 21y - 56 = 0$	Simplify
$2x + 3y - 8 = 0$	Divide both sides by 7

Sheet (5)

1. Find the vector equation of the straight line which passes through the origin point and the two straight lines whose equation $x = 3$, $y = 4$

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2. Find the vector equation of the straight line which passes through the point (3 , 1), and the point of intersection of the two lines whose equations $3x + 2y - 7 = 0$, $x + 3y = 7$ ……………

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3. Find the equation of the straight line passes through the point of intersection of the two straight lines whose equations $\vec{r} = k(-3 , 2)$, $3x - 2y = 13$ and parallel to the y-axis. ……

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4. Find the equation of the straight line passes through the point of intersection of the two lines whose equations $2x + y = 5$, $x + 5y = 16$ and perpendicular to the line whose equation $x - y = 8$ ……………………………………………

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Notes

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